Midterm

\[ \begin{align*}
\geq 90 & : 3 \\
80's & : 6 \\
70's & : 4 \\
60's & : 7 \\
50's & : 12 \\
40's & : 9 \\
<40 & : 5
\end{align*} \]

\[ \text{Median: 57} \]
\[ \text{Mean: 60} \]

Jums. ucsd.edu - Lean Theorem proven

**Completeness Theorem:**

(a) If \( T \) is consistent, then \( T \) is satisfiable

(b) If \( T \vdash A \), then \( T \vdash A \)

Showed already; (b) \( \implies \) (a).

**Lindenbaum's Theorem:** If \( T \) is consistent,
then there is a complete and consistent \( T \supset T' \).

**Defn:** \( T \) is complete if \( \forall A, A \in T \) or \( \neg A \in T \).

(Could have used: either \( T \vdash A \) or \( T \vdash \neg A \).)
Need to show there is a truth assignment \( \tau \) that satisfies \( T \), and hence \( T \).

**Lemma:** Suppose \( T \) is consistent and complete.

1. For any \( A \), \( A \in T \) iff \( \neg A \notin T \)
2. For any \( A, B \): \( A \rightarrow B \in T \) iff \( A \notin T \) or \( B \in T \)

**Pf.:** Proof of part (1). By completeness property, \( A \) and \( \neg A \) are not both in \( T \).

By "complete" property, at least one of \( A, \neg A \) is in \( T \). \( \square \)

**Proof of (2):**

Assume \( A \rightarrow B \in T \). Want to show \( A \notin T \) or \( B \in T \).

Assume also \( A \in T \) and \( B \notin T \) (for sake of a contradiction).

By (1), \( A \rightarrow B \in T \). So if \( A \in T, A \rightarrow B \in T \) - but this is inconsistent.

Assume \( A \notin T \) and \( A \rightarrow B \in T \). (for sake of a contradiction)

By (1), \( A \rightarrow B \in T \). Similarly, \( \neg A \in T \).

But \( \{ \neg A, A \rightarrow B \} \) is inconsistent since \( \neg A \land A \rightarrow B \)

Assume \( B \in T \) and \( A \rightarrow B \in T \). By (1), \( A \rightarrow B \in T \).

and \( \{ B, A \rightarrow B \} \) is inconsistent since \( B \land A \rightarrow B \)

qed Lemma.
Given $T$ is consistent and complete.
Want to show: exists $\varphi$ satisfying $T$.

Set $\varphi(p_i) = \begin{cases} T & \text{if } p_i \in T \\ F & \text{if } p_i \notin T \end{cases}$

Claim: $\varphi(A) = T$ iff $A \in T$, for all formulas $A$.

Pf by on the complexity of $A$.

Base case: $A$ is a propositional variable $p_i$.
Claim holds in this case directly from the definition of $\varphi$.

Ind Step #1: $A$ is $\neg B$.

$\begin{align*}
\varphi(A) = T & \iff \varphi(B) = F \\
& \iff B \notin T \\
& \iff A \in T
\end{align*}$

by defn of truth

by induction hypothesis

by Lemma (11)

Ind Step #2: $A$ is $(B \to C)$.

$\begin{align*}
\varphi(A) = T & \implies \varphi(B) = F \lor \varphi(C) = T \\
& \implies B \notin T \lor C \in T \\
& \implies B \to C \in T \\
& \implies A \in T
\end{align*}$

by defn of truth

by the two induction hypotheses

by Lemma (20)

by choice of $A, B, C$

\[ \square \quad \text{QED} \quad \text{Completeness Theorem!} \]
First-order logic:

Extend propositional logic.
Keep \( \neg, \land, \lor, \rightarrow, \leftrightarrow \).
Discard \( 
\neg \neg \).

Add variables \( x, y, z, \ldots \), range of \( \in \) domain universe of objects (aka individuals).

Predicates (aka, relations) such as \( \leq \), take object as input & return True/False.

Functions such as \( +, \cdot \),
Input to functions is individual, output is an individual.

Constants - Names for particular objects. 0

Quantifiers \( \forall, \exists \), \( \forall x (\ldots) \exists x (\ldots) \).
Examples

John - constant

Dog(x) - "x is a dog"
Cat(x) - "x is a cat"
Person(x) - "x is a person"

Likes(x, y) - "x likes y"
Mother(x) = mother of x (an individual)

\[ \forall x (\neg \text{Dog}(x) \lor \neg \text{Cat}(x)) \quad "\text{No dog is a cat}" \]

\[ \forall x (\neg (\text{Dog}(x) \land \text{Cat}(x))) \quad "\text{No dog is a cat}" \]

\[ \forall x \forall y (\text{Dog}(x) \land \text{Cat}(y) \rightarrow x \neq y) \]

\[ a \neq b \quad \neg (a \neq b) \quad a = b \quad \text{mean} \quad \neg (a = b) \]

= - special predicate symbol for "true" equality.

\[ \forall x (\text{Likes}(\text{John}, x) \rightarrow \text{Cat}(x)) \quad "\text{John likes only cats."} \]

John doesn't like anything except cats

- Each thing John likes is a cat.

\[ \forall \text{forall} \quad "\text{All}" \]
"John likes all cats"

\[ \forall x \ (\text{Liker}(\text{John}, x)) \implies \text{John likes everything} \]

\[ \forall x \ (\text{Cat}(x) \implies \text{Liker}(\text{John}, x)) \]

\[ \neg \exists x \ (\text{Cat}(x) \land \neg \text{Liker}(\text{John}, x)) \]

\[ \neg \exists x \ (\text{Cat}(x) \implies \neg \text{Liker}(\text{John}, x)) \]

\[ \forall A \ (\forall x \ A \iff \exists x \neg A) \]

"John likes some cat"

\[ \exists x \ (\text{Cat}(x) \land \text{Liker}(\text{John}, x)) \]

\[ \exists x \ (\text{Cat}(x) \rightarrow \text{Liker}(\text{John}, x)) - \text{true as long there is some } x \text{ that is not a cat or something John likes} \]

\[ \exists x \ (\neg \text{Cat}(x) \land \text{Liker}(\text{John}, x)) \]
For all reals $x$ (\ldots)

$\forall x \in \mathbb{R}$ (\ldots)

"bounded quantifier"

For some real $x$ (\ldots)

$\exists x \in \mathbb{R}$ (\ldots)

$\exists x (x \in \mathbb{R} \land \ldots)$