**Math 20F - Homeworks 9 & 10 - Selected answers**

Samuel R. Buss - Winter 2003 - UC San Diego

Revision 1.0. – March 13, 2003

**Section 6.1, Problem 19.** We did almost exactly this problem as a theorem proved in class.

**Section 6.1, Problem 22.** Since \( u_j^T u_i = \delta_{i,j} \), we have

\[
Au_i = \sum_{j=1}^{n} c_j u_j^T u_i = \sum_{j=1}^{n} \delta_{i,j} c_j u_j = c_i u_i.
\]

Hence \( u_i \) is an eigenvector for the eigenvalue \( c_i \).

**Section 6.3, Problem 1(a).** We did this as an example in class on Wednesday.

**Section 6.3, Problem 1(c).** \( \det(A - \lambda I) = (2 - \lambda)(-4 - \lambda) + 8 = 2\lambda + \lambda^2 = (2 + \lambda)(\lambda) \). The roots of the characteristic polynomial are \( \lambda_1 = 1 \) and \( \lambda_2 = -2 \): these are the eigenvalues of \( A \). Solving \( Ax = 0 \) for a nontrivial \( x \), we find that \( x_1 = (4, 1)^T \) is an eigenvector corresponding to the eigenvalue \( \lambda_1 = 0 \). Solving \( (A + 2I)x = 0 \), we find that \( x_2 = (2, 1)^T \) is an eigenvector corresponding to the eigenvalue \( \lambda_2 = -2 \).

The eigenvalues are distinct, hence \( x_1 \) and \( x_2 \) are linearly independent. Therefore, \( A \) is diagonalizable in the following form:

\[
A = \begin{pmatrix}
4 & 2 \\
1 & 1 \\
\end{pmatrix} \cdot \begin{pmatrix}
0 & 0 \\
0 & -2 \\
\end{pmatrix} \cdot \begin{pmatrix}
4 & 2 \\
1 & 1 \\
\end{pmatrix}^{-1}
\]

\[
= \begin{pmatrix}
4 & 2 \\
1 & 1 \\
\end{pmatrix} \cdot \begin{pmatrix}
0 & 0 \\
0 & -2 \\
\end{pmatrix} \cdot \begin{pmatrix}
1/2 & -1 \\
-1/2 & 2 \\
\end{pmatrix}
\]

**Section 6.3, Problem 3(c).** The sixth power of \( A \) is equal to

\[
A^6 = \begin{pmatrix}
4 & 2 \\
1 & 1 \\
\end{pmatrix} \cdot \begin{pmatrix}
0 & 0 \\
0 & -2 \\
\end{pmatrix}^6 \cdot \begin{pmatrix}
1/2 & -1 \\
-1/2 & 2 \\
\end{pmatrix}
\]

This can be multiplied out by hand if desired – start by computing the sixth power of the diagonal matrix. This particular case is rather easy since there is only one non-zero eigenvariable. You will find that

\[
A^6 = \begin{pmatrix}
-64 & 256 \\
-32 & 128 \\
\end{pmatrix}.
\]
Section 6.3, Problem 4. The idea behind these problems is as follows. First diagonalize the matrix $A$ as $A = SDS^{-1}$. If the eigenvalues are non-negative, the diagonal matrix $D$ has non-negative entries along the diagonal. A matrix $E$ such that $E^2 = D$ can be formed by letting $E$ be the diagonal matrix whose entries are the square roots of the entries of $D$. Then, letting $B = SES^{-1}$, we have $B^2 = A$.

Section 6.3, Problem 8(a), 9. I neglected to define “defective” in class Wednesday (although I intended to). Therefore, as I promised, this term will not appear on the final exam. An $n \times n$ matrix that does not have $n$ linearly independent eigenvectors is called defective. That is to say, a matrix is diagonalizable if and only if it is not defective.

Section 6.3, Problem 9. If $A$ has one eigenvalue (call it $\lambda_1$) of multiplicity 3, then the other eigenvalue (call it $\lambda_2$) has multiplicity 1. Now, there is an eigenvector $x_2$ for $\lambda_2$ of course. Furthermore, since $\text{rank}(A - \lambda_1 I)$ is equal to 1, then the null space of $A - \lambda_1 I$ has dimension 3; therefore, there are three linearly independent eigenvectors for $\lambda_1$. Further, since $\lambda_1 \neq \lambda_2$ (they are unequal, since otherwise, the eigenvalue would have multiplicity four!), $x_2$ is not in the eigenspace of $\lambda_1$. Thus, the four eigenvectors are linearly independent, so $A$ is diagonalizable and $A$ is not defective.