Math 260B — Mathematical Logic — Winter 2020 — U.C. San Diego
Instructor: Sam Buss

HOMEWORK ASSIGNMENTS
Hand in on paper or via email

1. Prove that a set S (of integers) is r.e. (recursively enumerable) iff is the domain of a recursive function. (For this and the next problem is it OK to work Turing machines and/or with intuitive notions of computation, i.e., to invoke the Church-Turing thesis.)

2. Prove that a set S (of integers) is r.e. iff is the range of a recursive function.

3. Prove that Q does not prove $\forall x(0 + x = x)$.

Problems 4.-11.: Suggest to do only one, and post to piazza for others to verify. Hints are given on pp. 85-86 of the *Handbook of Proof Theory*.

4. Show that $I\Delta_0$ proves multiplication is commutative.

5. Show that $I\Delta_0$ proves multiplication is associative.

6. Show that $I\Delta_0$ proves the distributive law for $+$ and $\cdot$.

7. Show that $I\Delta_0$ proves the cancellation law for addition:
$\forall x, y, z(x + z = y + z \to x = y)$.

8. Show that $I\Delta_0$ proves transitivity of $\leqslant$.

9. Show that $I\Delta_0$ proves $\forall x, y(x + y = 0 \to x = 0 \vee y = 0)$ and
$\forall x, y(x \cdot y = 0 \to x = 0 \vee y = 0)$. (Anti-idempotency)

10. Show that $I\Delta_0$ proves reflexivity, tricotomy, and antisymmetry of $\leqslant$.

11. Show that $I\Delta_0$ proves the cancellation law for multiplication:
$\forall x, y, z(x \cdot Sz = y \cdot Sz \to x = y)$.

12. Give an example of a consistent, axiomatizable theory $T \supset I\Delta_0$ which is not ω-consistent.

13. Prove that if T is not ω-consistent, then T is not a subtheory of $Th(\mathbb{N})$; in other words, T proves some false sentence.

14. Prove Löb's Theorem: Suppose $T \supset I\Delta_0$ is consistent and axiomatizable and satisfies the Hilbert-Bernay-Löb conditions. Further suppose φ is a sentence such that T proves $Thm_T(\ulcorner\varphi\urcorner) \to \varphi$. Then $T \vdash \varphi$.

15.★ Examine the AKS polynomial algorithm and determine if it can be formalized in S^1_2 or $I\Delta_0 + \Omega_1$. (The former theory is for polynomial time; the latter for the polynomial time hierarchy.)