Math260 - Introduction to Mathematical Logic

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Inference Rules for Natural Deduction.

Propositional rules are below. First-order logic rules are on the second page.

$$\wedge i\colon \frac{\varphi \qquad \psi}{\varphi \wedge \psi} \qquad\qquad \wedge e_1\colon \frac{\varphi \wedge \psi}{\varphi} \qquad\qquad \wedge e_2\colon \frac{\varphi \wedge \psi}{\psi}$$

$$\vee i_1\colon \frac{\varphi}{\varphi \vee \psi} \qquad\qquad \vee i_2\colon \frac{\psi}{\varphi \vee \psi} \qquad\qquad \vee e\colon \frac{\varphi \vee \psi \qquad \begin{array}{c}[\varphi]\\ \vdots \\ \chi\end{array} \qquad \begin{array}{c}[\psi]\\ \vdots \\ \chi\end{array}}{\chi}$$

$$\rightarrow i\colon \frac{\begin{array}{c}[\varphi]\\ \vdots \\ \psi\end{array}}{\varphi \rightarrow \psi} \qquad\qquad \rightarrow e\colon \frac{\varphi \qquad \varphi \rightarrow \psi}{\varphi} \text{ Modus Ponens}$$

$$\perp e\colon \frac{\perp}{\psi} \qquad\qquad \text{Reductio Ad Absurdum: } \frac{\begin{array}{c}[\varphi \rightarrow \perp]\\ \vdots \\ \perp\end{array}}{\varphi}$$

Note that there are no rules for negation ($\neg$). One could add two rules of the form $\frac{\neg\neg\varphi}{\varphi}$ and $\frac{\varphi}{\neg\neg\varphi}$. However, instead, the convention is that $\neg\varphi$ is an abbreviation for $(\varphi \rightarrow \perp)$. The two double-negation rules are then derived rules.

Other derived rules include:

$$\text{Modus Tollens: } \frac{\varphi \rightarrow \psi \qquad \neg\psi}{\neg\varphi} \qquad\qquad \text{Cut: } \frac{\varphi \rightarrow \psi \qquad \psi \rightarrow \chi}{\varphi \rightarrow \chi}$$

$$\text{Law of the Excluded Middle: } \frac{}{\varphi \vee \neg\varphi}$$

The above rules are for classical logic. If the RAA rule is omitted, then intuitionistic logic is obtained. If the $\perp e$ inference rule is also omitted, then minimal logic is obtained.

Additional rules for first-order logic:

Whenever a formula $\varphi(t/x)$ is written, it is implicitly understood that t must be substitutable for x in φ.

Rules for equality:

$$= i\colon \frac{}{t = t} \qquad\qquad = e\colon \frac{s = t \qquad \varphi(s/x)}{\varphi(t/x)}$$

Rules for universal quantification:

$$\forall i\colon \frac{\varphi}{\forall x \varphi} \qquad\qquad \forall e\colon \frac{\forall x \varphi}{\varphi(t/x)}$$

where, for the $\forall i$ inference, the *eigenvariable condition* must hold: namely, there is no free occurence of x in any undischarged formula in the subproof above the hypothesis φ.

Rules for existential quantification:

$$\exists i : \frac{\varphi(t/x)}{\exists x \varphi} \qquad\qquad \forall e\colon \frac{\exists x \varphi \qquad \begin{array}{c}[\varphi(y/x)]\\ \vdots \\ \psi\end{array}}{\psi}$$

where, for the $\exists e$ inference, the *eigenvariable condition* must hold: namely, (a) y does not occur free in $\exists x \varphi$ or in ψ and (b) in the proof above the right hypothesis ψ, the variable y does not have any free occurences in any undischarged hypothesis other than the indicated occurences in $\varphi(y/x)$.