

Math260 - Introduction to Mathematical Logic

Fall 2007 – Winter 2008

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Homework #3. Due Thursday, November 8, 2007.

Definition. Let Γ and Π be sets of formulas. We define $\Gamma \models \Pi$ as $\forall \varphi \in \Pi, \Gamma \models \varphi$.

1. Suppose Γ is finite. Prove that there is a formula φ such that $\varphi \models \Gamma$ and $\Gamma \models \varphi$. [Hint: Don't make this problem harder than it needs to be.]
2. Suppose that Γ is infinite and Π is finite. Further suppose $\Gamma \models \Pi$ and $\Pi \models \Gamma$. Prove that there is a finite $\Gamma_0 \subset \Gamma$ such that $\Gamma_0 \models \Gamma$.
3. Suppose that Γ and Π have the property that for any truth assignment τ , exactly one of $\tau \models \Gamma$ or $\tau \models \Pi$ holds. Prove that there is a finite Δ such that $\Gamma \models \Delta$ and $\Delta \models \Gamma$.
4. Using the predicate symbols $L(x, y)$ for “ x loves y ”, and $M(x, y)$ for “ x is the mother of y ”, translate the following sentences into first-order logic. Use as simple a formula as possible, and use equality only if necessary.
 - a. Everyone has a mother.
 - b. Everyone has exactly one mother.
 - c. Every mother loves all her children.
 - d. No one loves everyone.
 - e. If someone loves everyone, then they are loved by everyone.
 - f. If someone loves everyone else, then they are loved by everyone else.
 - g. Abe loves everyone who does not love their mother.
 - h. Abe loves only people who love themselves.
5. Give natural deduction proofs of the following five formulas.
 - a. $\forall x P(x) \vdash \exists x P(x)$.
 - b. $\forall x (P(x) \rightarrow Q(x)), \exists x P(x) \vdash \exists x Q(x)$.

- c. $\exists x(P(x) \rightarrow Q(x)), \forall xP(x) \vdash \exists xQ(x)$.
 - d. $\exists y\forall xP(x, y) \vdash \forall x\exists yP(x, y)$.
 - e. $\exists x(P(x) \wedge \varphi) \vdash \exists xP(x) \wedge \varphi$, provided x does not occur free in φ .
- 6.** Give an example of how it is possible to derive a non-valid formula if the eigenvariable condition of the $\exists e$ inference rule does not hold.
- 7.** Give a natural deduction proof of $\exists x(P(x) \rightarrow \forall yP(y))$.
- 8.** Give natural deduction proofs of:
- a. $\forall x\varphi \vdash \neg\exists x\neg\varphi$.
 - b. $\neg\exists x\neg\varphi \vdash \forall x\varphi$.
- 9.** Give natural deduction proofs of:
- a. $\forall x\varphi(x) \vdash \forall y\varphi(y)$.
 - b. $\exists x\varphi(x) \vdash \exists y\varphi(y)$.
- 10.** Give natural deduction proofs of:
- a. $r = s, P(r) \vdash P(s)$.
 - b. $r = s, t = u \vdash f(r, t) = f(s, u)$.
 - c. $r = s, s = t \vdash r = t$.
 - d. $x \neq y \vdash \forall z(z \neq x \vee z \neq y)$.
 - e. $\exists x\exists y(x \neq y) \vdash \forall x\exists y(x \neq y)$.
 - f. $\forall x\exists y(f(y) = x) \vdash \forall x\exists y(f(f(y)) = x)$.