

# Math260 - Introduction to Mathematical Logic

Fall 2007 – Winter 2008

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Homework #4. Due Thursday, November 29, 2007.

## Definition.

1. Which of the following terms are substitutable as stated?

- $x$  for  $x$  in  $x = x$ .
- $y$  for  $x$  in  $x = x$ .
- $x$  for  $y$  in  $x = x$ .
- $0 + x$  for  $x$  in  $\exists x(y = x)$ .
- $0 + y$  for  $x$  in  $\exists x(y = x)$ .
- $0 + x$  for  $y$  in  $\exists x(y = x)$ .
- $0 + y$  for  $y$  in  $\exists x(y = x)$ .

2. For each of the following, state whether the statement is true or false. If false, prove the falsity by giving an explicit example of a model  $\mathcal{M}$  and an object assignment  $\ell$  (and formulas  $\varphi$  and  $\psi$ ) that makes the statement false.

- $\models \forall x(P(x) \rightarrow \forall xP(x))$ .
- $\models \exists x(P(x) \rightarrow \forall xP(x))$ .
- $\forall x\varphi \models \exists x\varphi$ .
- $\exists x\varphi \models \forall x\varphi$ .
- $(\exists x\varphi \rightarrow \exists x\psi) \models \exists x(\varphi \rightarrow \psi)$ .
- $\exists x(\varphi \rightarrow \psi) \models (\exists x\varphi \rightarrow \exists x\psi)$ .

3. Let  $\varphi$  and  $\psi$  be sentences. Consider the two statements

$$(\alpha) \quad \mathcal{M} \models \varphi \Rightarrow \mathcal{M} \models \psi \text{ for all models } \mathcal{M}.$$

and

$$(\beta) \quad \models \varphi \Rightarrow \models \psi.$$

Prove that  $(\alpha)$  implies  $(\beta)$ . Prove that the converse does not hold by giving an explicit example.

4. Let  $L$  be a language and  $c$  a new constant symbol not in  $L$ . Let  $\Gamma$  be a set of  $L$ -formulas, and let  $\Delta$  be  $\Gamma \cup \{\exists x\varphi \rightarrow \varphi(c/x)\}$ , for some  $L$ -formula  $\varphi$ .

Prove that  $\Delta$  is conservative over  $\Gamma$  w.r.t.  $L$ -formulas. That is, show that if  $\psi$  is an  $L$ -formula and  $\Delta \vdash \psi$ , then  $\Gamma \vdash \psi$ . (Hint: we proved a related result in class as part of the completeness theorem's proof.)

5. One way to try to express the pigeonhole principle is by the formula PHP defined as

$$\exists z\forall x(f(x) \neq z) \rightarrow \exists x\exists y(f(x) = f(y) \wedge x \neq y).$$

- a. Express in English what PHP is saying.
- b. Prove that  $\not\models PHP$ .
- c. In what situations (what kinds of models) does PHP actually correctly express the pigeonhole principle?