Math260 - Introduction to Mathematical Logic

Fall 2007 – Winter 2008

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Homework #5. Due Friday, December 14, 2007.

1. Let T be $\{\forall x \exists y(f(y) = x),\ \exists y \forall z(P(f(z)) \leftrightarrow \neg P(y))\}$.

a. Form the Skolemization T^S of T.

b. What are the **clauses** that are obtainable from T^S and the equality axoms? These should be the clauses that serve as input to a Robinson-resolution refutation.

c. Give a Robinson-style refutation, indicating explicitly the mgu's (most general unifiers) used.

2. Let T be a set of sentences over a language that has equality as a logical symbol. Let T' be the same set of sentences, but over a language where equality is a non-logical symbol.

a. Prove that T is satisfiable, then T' is satisfiable.

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Hint: a. has an easier construction than b. In both cases, the idea is to transform a model for one theory into a model for the other.

3. We define $\mathcal{N}$ to be a *substructure* of $\mathcal{M}$, provided that $|\mathcal{N}| \subseteq |\mathcal{M}|$ and that the interpretations $c^{\mathcal{N}}$, $P^{\mathcal{N}}$ and $f^{\mathcal{N}}$ are the same as the interpretations for $\mathcal{M}$, but restricted to the universe of $\mathcal{N}$. Thus, $c^{\mathcal{M}} \in |\mathcal{N}|$, and $|\mathcal{N}|$ is closed under function symbols $f^{\mathcal{M}}$.

a. Let T be a universal theory. Prove that if $\mathcal{N}$ is a substructure of $\mathcal{M}$ and $\mathcal{M} \vDash T$, then $\mathcal{N} \vDash T$. (When this latter property holds, we say T is *preserved under substructures.*)

b. Prove the converse holds too. That is, suppose that T is a set of sentences and is preserved under substructures. Prove that T is a universal theory. [This is a fairly hard problem, even with the following hint: Consider the set S of universal sentences logically implied by T. Suppose $\mathcal{M}$ is a model of S. It suffices to find a model $\mathcal{N}$ of T such that $\mathcal{M}$ is a substructure of $\mathcal{N}$. (Why?).]