Math260AB - Set Theory
Fall 2012 – Winter 2013
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Handout 1: Axioms for Set Theory

1 The First-order Language of Set Theory

The language of set theory contains the following basic symbols.

Boolean connectives: $\neg$, $\rightarrow$, $\wedge$, $\vee$, $\leftrightarrow$.

Quantifiers: $\forall$ and $\exists$

Variables: $x_1, x_2, x_3, \ldots,$, also denoted $x, y, z, \ldots$. Variables range over the universe of all sets.

Equality: $=$

Membership: $\in$

Parentheses: (,)

All (or nearly all) set-theoretic assertions we will ever make are expressible in the above first-order language. Of course, we will introduce many abbreviations to avoid having to write very long statements in the base first-order language.

2 Axioms of Set Theory

Some useful abbreviations include:

$$
\begin{array}{rcccl}
\exists! x\varphi(x) & \iff & \exists x[\varphi(x) \wedge \forall y(\varphi(y) \rightarrow x = y)] & & \\
\exists x{\in}y\,\varphi & \iff & \exists x(x \in y \wedge \varphi) & & \\
\forall x{\in}y\,\varphi & \iff & \forall x(x \in y \rightarrow \varphi) & & \\
x \notin y & \iff & \neg x \in y & & \\
x \subseteq y & \iff & \forall z(z \in x \rightarrow z \in y) & & \\
x = \emptyset & \iff & \forall z(z \notin x) & & \\
\emptyset \in x & \iff & \forall z(z = \emptyset \rightarrow z \in x) & \iff & \exists z(z = \emptyset \wedge z \in x) \\
y = S(x) & \iff & \forall z(z \in y \leftrightarrow z = x \vee z \in x) & & \text{“}y\text{ is the successor of }x\text{”} \\
S(y) \in x & \iff & \forall z(z = S(y) \rightarrow z \in x) & \iff & \exists z(z = S(y) \wedge z \in x) \\
y \in x \cap z & \iff & y \in x \wedge y \in z & & \\
y = x \cap z & \iff & \forall w(w \in y \leftrightarrow w \in x \wedge w \in z) & & \\
\mathrm{SING}(x) & \iff & (\exists y \in x)(\forall z \in x)(y = z) & & (\text{“}x\text{ is a singleton})
\end{array}
$$

The axioms for Zermelo-Fraenkel set theory with Choice, *ZFC*, are:

0. Logical Axioms and Equality Axioms: These are axioms that state the basic properties of the Boolean connectives, the quantifiers, and the equality symbol. This includes the assertion that some set exists.

1. Extensionality: A set is determined by its members:

$$\forall x \forall y [\forall w (w \in x \leftrightarrow w \in y) \rightarrow x = y].$$

2. Foundation (Regularity): Every non-empty set contains an $\in$-minimal element:

$$\forall x [\exists y (y \in x) \rightarrow \exists y (y \in x \wedge \neg \exists z (z \in x \wedge z \in y)].$$

3. Comprehension Axioms (Separation): For every first-order formula $\varphi(x, \vec{z})$ that does not involve y,

$$\forall \vec{z} \forall v \exists y \forall x (x \in y \leftrightarrow x \in v \wedge \varphi(x, \vec{z})).$$

Note there are infinitely many comprehension axioms!

4. Pairing

$$\forall x \forall y \exists z (x \in z \wedge y \in z).$$

5. Union:

$$\forall x \exists z \forall y \forall w (w \in y \wedge y \in x \rightarrow w \in z).$$

6. Replacement Axioms: The image of a first-order definable function on a set w exists. Let $\varphi(x, z, \vec{y})$ be a first-order formula.

$$\forall \vec{y} \forall w [((\forall x {\in} w) \exists ! z \varphi(x, z, \vec{y})) \rightarrow \exists s ((\forall x {\in} w)(\exists z {\in} s) \varphi(x, z, \vec{y}))].$$

There are also infinitely many instances of the replacement axiom.

7. Infinity: There exists an infinite set:

$$\exists x (0 \in x \wedge \forall y (y \in x \rightarrow S(y) \in x)).$$

8. Power Set Axiom:

$$\forall x \exists z \forall w (w \subseteq x \rightarrow w \in z).$$

9. Axiom of Choice (AC) A set F of disjoint nonempty sets has a set C of distinct representatives.

$$\forall F [\emptyset \notin F \wedge \forall x {\in} F \forall y {\in} F (x \neq y \rightarrow x \cap y = \emptyset) \rightarrow \exists C \forall x {\in} F (\mathrm{SING}(C \cap x))].$$