1 The First-order Language of Set Theory

The language of set theory contains the following basic symbols.

Boolean connectives: \( \neg, \rightarrow, \land, \lor, \leftrightarrow \).

Quantifiers: \( \forall \) and \( \exists \)

Variables: \( x_1, x_2, x_3, \ldots \), also denoted \( x, y, z, \ldots \). Variables range over the universe of all sets.

Equality: \( = \)

Membership: \( \in \)

Parentheses: (,),

All (or nearly all) set-theoretic assertions we will ever make are expressible in the above first-order language. Of course, we will introduce many abbreviations to avoid having to write very long statements in the base first-order language.

2 Axioms of Set Theory

Some useful abbreviations include:

- \( \exists! x \varphi(x) \iff \exists x[\varphi(x) \land \forall y(\varphi(y) \rightarrow x = y)] \)
- \( \exists x \in y \varphi \iff \exists x(x \in y \land \varphi) \)
- \( \forall x \in y \varphi \iff \forall x(x \in y \rightarrow \varphi) \)
- \( x \notin y \iff \neg x \in y \)
- \( x \subseteq y \iff \forall z(z \in x \rightarrow z \in y) \)
- \( x = \emptyset \iff \forall z(z \notin x) \)
- \( \emptyset \in x \iff \forall z(z = \emptyset \rightarrow z \in x) \iff \exists z(z = \emptyset \land z \in x) \)
- \( y = S(x) \iff \forall z(z \in y \leftrightarrow z = x \lor z \in x) \) “\( y \) is the successor of \( x \)”
- \( S(y) \in x \iff \forall z(z = S(y) \rightarrow z \in x) \iff \exists z(z = S(y) \land z \in x) \)
- \( y \in x \cap z \iff y \in x \land y \in z \)
- \( y = x \cap z \iff \forall w(w \in y \leftrightarrow w \in x \land w \in z) \)
- \( \text{SING}(x) \iff (\exists y \in x)(\forall z \in x)(y = z) \) (“\( x \) is a singleton”)
The axioms for Zermelo-Fraenkel set theory with Choice, ZFC, are:

0. **Logical Axioms and Equality Axioms:** These are axioms that state
   the basic properties of the Boolean connectives, the quantifiers, and
   the equality symbol. This includes the assertion that some set exists.

1. **Extensionality:** A set is determined by its members:
   \[ \forall x \forall y [\forall w (w \in x \leftrightarrow w \in y) \rightarrow x = y]. \]

2. **Foundation (Regularity):** Every non-empty set contains an \( \in \)-minimal
   element:
   \[ \forall x [\exists y (y \in x) \rightarrow \exists y (y \in x \land \neg \exists z (z \in x \land z \in y))]. \]

3. **Comprehension Axioms (Separation):** For every first-order formula
   \( \varphi(x, z) \) that does not involve \( y \),
   \[ \forall \exists v \exists y \forall x (x \in y \leftrightarrow x \in v \land \varphi(x, z)). \]
   Note there are infinitely many comprehension axioms!

4. **Pairing**
   \[ \forall x \forall y \exists z (x \in z \land y \in z). \]

5. **Union:**
   \[ \forall x \exists z \forall y w (w \in y \land y \in x \rightarrow w \in z). \]

6. **Replacement Axioms:** The image of a first-order definable function
   on a set \( w \) exists. Let \( \varphi(x, z, \bar{y}) \) be a first-order formula.
   \[ \forall y \forall w [((\forall x \in w) \exists ! z \varphi(x, z, \bar{y})) \rightarrow \exists s ((\forall x \in w) (\exists z \in s) \varphi(x, z, \bar{y}))]. \]
   There are also infinitely many instances of the replacement axiom.

7. **Infinity:** There exists an infinite set:
   \[ \exists x (0 \in x \land \forall y (y \in x \rightarrow S(y) \in x)). \]

8. **Power Set Axiom:**
   \[ \forall x \exists z \forall w (w \subseteq x \rightarrow w \in z). \]

9. **Axiom of Choice (AC)** A set \( F \) of disjoint nonempty sets has a set \( C \)
    of distinct representatives.
   \[ \forall F [\emptyset \notin F \land \forall x \in F \forall y \in F (x \neq y \rightarrow x \cap y = \emptyset) \rightarrow \exists C \forall x \in F (\text{SING}(C \cap x))]. \]