The Completeness Theorem for LK

We begin the proof of the Completeness Theorem for the first-order sequent calculus. Note that our proof only applies for countable languages without equality.

**Completeness Theorem 1** Let $\Gamma \rightarrow \Delta$ be a sequent in a first-order language $L$ which does not contain equality. If $\mathcal{S}$ is a set of sequents and $\forall \mathcal{S} \models \Gamma \rightarrow \Delta$, then there exists a finite subset of $\forall \mathcal{S}$, $\Pi$, such that $\Pi, \Gamma \rightarrow \Delta$ has an LK proof.

We also provide an alternate formulation:

**Completeness Theorem 2** Let $\Gamma \rightarrow \Delta$ be a sequent in a first-order language $L$ which does not contain equality. If $T$ is a set of sentences and $T \models \Gamma \rightarrow \Delta$, then there exists a finite $\Pi \subseteq T$ such that $\Pi, \Gamma \rightarrow \Delta$ has an LK proof.

The idea of our proof will be to work backwards to find of proof of $\Pi, \Gamma \rightarrow \Delta$ (we do not yet know what this $\Pi$ will be, however).

We begin by enumerating all $L$-formulas as

$$A_1, A_2, A_3, \ldots$$

where every $L$-formula appears infinitely often. We can do this as follows.

Since we assume $L$ to be countable, enumerate the functions, predicates, and constants in $L$:

$$f_1, f_2, f_3, \ldots$$
$$P_1, P_2, P_3, \ldots$$
$$c_1, c_2, c_3, \ldots$$
Now, for $i = 1, 2, 3, \ldots$ list out all $L$-formulas with $\leq i$ symbols that have subscripts $\leq i$. This enumeration will guarantee that we list all $L$-formulas and that each $L$-formula will appear over and over again.

Likewise, enumerate all $L$-terms $t_1, t_2, t_3, \ldots$.

And then enumerate all formula-term pairs, $< A_i, t_j >$, in a list where again each pair appears infinitely often. For example, we can use a loop similar to the one defined above:

$< A_1, t_1 >, < A_1, t_1 >, < A_1, t_1 >, < A_2, t_2 >, < A_2, t_2 >, \ldots$.

Now we try building a proof $P$. Start with $\Gamma \rightarrow \Delta$. If $\Gamma \cap \Delta \neq \emptyset$, then it is easy to give a proof of $\Gamma \rightarrow \Delta$ using $\text{Weakening: left}$ and $\text{Weakening: right}$ inferences. This motivates the following definition.

**Definition** The sequent $\Gamma' \rightarrow \Delta'$ is **active** if $\Gamma' \cap \Delta' = \emptyset$.

As we build $P$, we will work on active sequents. Note that $P$ will be a tree of sequents.

To begin with, $P$ will be the single sequent $\Gamma \rightarrow \Delta$. Take the next pair$^1$ $< A, t >$ in the enumeration.

**Step 1:** If $A \in \forall S$ (or if $A \in T$ depending on which of the above formulations one prefers), add $A$ to every antecedent in $P$. $\Pi$ will end up being the set of such $A$’s added by this step.$^2$

**Step 2:** For every active sequent that contains $A$, update it as follows:

**Case a):** If $A$ is $\neg B$ and a sequent $\neg B, \Gamma' \rightarrow \Delta'$ is active in $P$, replace it by:

$$
\begin{array}{c}
\neg B, \Gamma' \rightarrow \Delta', B \\
\hline
\neg B, \Gamma' \rightarrow \Delta'
\end{array}
$$

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$^1$The *first* next pair is the first pair in the enumeration.

$^2$Since $T$ contains sentences, $A$ is a sentence so we do not have to worry about contradicting eigenvariable conditions. The same holds if one prefers the $\forall S$ formulation.
If $A$ is $\neg B$ and a sequent $\Gamma' \rightarrow \Delta'$, $\neg B$ is active in $P$, replace it by:

$$
\frac{B, \Gamma' \rightarrow \Delta', \neg B}{\Gamma' \rightarrow \Delta', \neg B}
$$

Note that the upper sequent could now be inactive, but it could also still be active.

**Case b):** If $A$ is $B \land C$, then any active sequent $B \land C, \Gamma' \rightarrow \Delta'$ in $P$ is replaced by:

$$
\frac{B, C, B \land C, \Gamma' \rightarrow \Delta'}{B \land C, \Gamma' \rightarrow \Delta'}
$$

If $A$ is $B \land C$, then any active sequent $\Gamma' \rightarrow \Delta', B \land C$ in $P$ is replaced by:

$$
\frac{\Gamma' \rightarrow \Delta', B \land C, B \land C, \Gamma' \rightarrow \Delta'}{\Gamma' \rightarrow \Delta', B \land C}
$$

Again, the upper sequents may be active or inactive.

**Case c):** If $A$ is $B \lor C$, then every active sequent in $P$ of the form $B \lor C, \Gamma' \rightarrow \Delta'$ is replaced by:

$$
\frac{B, B \lor C, \Gamma' \rightarrow \Delta' \quad C, B \lor C, \Gamma' \rightarrow \Delta'}{B \lor C, \Gamma' \rightarrow \Delta'}
$$

If $A$ is $B \lor C$, then every active sequent in $P$ of the form $\Gamma' \rightarrow \Delta', B \lor C$ is replaced by:

$$
\frac{\Gamma' \rightarrow \Delta', B \lor C, B, C}{\Gamma' \rightarrow \Delta', B \lor C}
$$

**Case d):** If $A$ is $B \rightarrow C$, then any active sequent in $P$ of the form $\Gamma' \rightarrow \Delta', B \rightarrow C$ is replaced by:

$$
\frac{B, \Gamma' \rightarrow \Delta', C, B \rightarrow C}{\Gamma' \rightarrow \Delta', B \rightarrow C}
$$

If $A$ is $B \rightarrow C$, then any active sequent in $P$ of the form $B \rightarrow C, \Gamma' \rightarrow \Delta'$ is replaced by:

$$
\frac{B \rightarrow C, \Gamma' \rightarrow \Delta', B \quad C, B \rightarrow C, \Gamma' \rightarrow \Delta'}{B \rightarrow C, \Gamma' \rightarrow \Delta'}
$$
Case e): Suppose $A$ is $\forall x B(x)$. Let $\Gamma' \rightarrow \Delta', \forall x B(x)$ be an active sequent, let $b$ be some (new) free variable, not used anywhere in $P$ yet. Then replace this with:

$$
\begin{align*}
\Gamma' & \rightarrow \Delta', \forall x B(x), B(b) \\
\Gamma' & \rightarrow \Delta', \forall x B(x)
\end{align*}
$$

Note that in this case the top cedent will certainly be active since $b$ is a completely new variable.

Likewise, if $\forall x B(x), \Gamma' \rightarrow \Delta'$ is active then replace it by:

$$
B(t), \forall x B(x), \Gamma' \rightarrow \Delta'
\Rightarrow
\forall x B(x), \Gamma' \rightarrow \Delta'
$$

Note that the term $t$ from our ordered pair $< A, t >$ finally comes into play here.

Case f): If $A$ is of the form $\exists x B(x)$, $c$ is a new free variable not used in $P$ yet, and $\exists x B(x), \Gamma' \rightarrow \Delta'$ is active in $P$, replace it by:

$$
B(c), \exists x B(x), \Gamma' \rightarrow \Delta'
\Rightarrow
\exists x B(x), \Gamma' \rightarrow \Delta'
$$

Likewise any active sequent of the form $\Gamma' \rightarrow \Delta', \exists x B(x)$ is replaced by:

$$
\Gamma' \rightarrow \Delta', \exists x B(x), B(t)
\Rightarrow
\Gamma' \rightarrow \Delta', \exists x B(x)
$$

The term $t$ is used in this case as well.

Clearly, if we are done are finitely many steps, then the last of $P$ is $\Pi, \Gamma \rightarrow \Delta$, and we are done. What if we do not halt after finitely many steps? We will show that the hypothesis of the Completeness Theorem fails, i.e.

$$
\forall S \not\models \Gamma \rightarrow \Delta, \text{ or in the other formulation,} \quad T \not\models \Gamma \rightarrow \Delta
$$

However, this construction will have to wait until next time.