1 (Failed) Proof: Completeness for Uncountable Languages

Let \( L \) be a language of cardinality \( \kappa \)
Let \( \Gamma \) be a set of \( L \)-sentences.

Either there is a model \( M \) of \( \Gamma \) or there exists a finite \( \Gamma_0 \subseteq \Gamma \) such that
\[ \Gamma_0 \rightarrow \]
has a proof.

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Cardinality of \( \Gamma \) is \( \leq \kappa \) (since the set of \( L \)-sentences has cardinality \( =\kappa \))

Enumerate \( \Gamma \) as a well-ordered sequence \( \gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_\kappa \)
Assume there is an inexhaustible supply of variables (need \( \kappa \) many)

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Form \( \Lambda, \Xi \) as before
\( \Lambda \subseteq \Gamma \)
\( \Lambda, \Xi \) satisfy closure properties (1)-(6)

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Do it in steps:
Definition:
\[ \Lambda_\alpha, \Xi_\alpha \leq \kappa \]
\[ \Lambda_{-1} = \Xi_{-1} = \emptyset \]
Assume \( \Lambda_\beta, \Xi_\beta \) have been defined for all \( \beta < \alpha \)

Construct \( \Lambda_\alpha, \Xi_\alpha \) such that there is no finite \( \Gamma_0 \subseteq \Lambda_\alpha \cup \Gamma, \Delta_0 \subseteq \Xi_\alpha \) such that \( \Gamma_0 \rightarrow \Delta_0 \) has a proof, and such that \( \Lambda_\alpha, \Xi_\alpha \) satisfy conditions (1)-(6), and \( \gamma_\beta \in \Lambda_\alpha \) for \( \beta \leq \alpha \).

To define \( \Lambda_\alpha, \Xi_\alpha \) work backwards from the sequent
\[ \gamma_\alpha \rightarrow \]
and try to give a proof.

Define "active":
\[ \Gamma' \rightarrow \Delta' \] is active if there is no finite \( \Gamma_0 \subseteq \bigcup_{\beta<\alpha} \Lambda_\beta \cup \Gamma, \Delta_0 \subseteq \bigcup_{\beta<\alpha} \Xi_\beta, \]
such that \( \Gamma_0, \Gamma' \rightarrow \Delta_0 \Delta' \) has a proof.

—
Form an "unproof" of \( \gamma_\alpha \rightarrow \) as before, which has an infinite branch of nonactive sequents. Let:
\[ \Lambda_\alpha = ( \bigcup_{\beta<\alpha} \Lambda_\beta ) \cup \{ \text{formulas in antecedents of this finite branch} \}, \]
\[ \Xi_\alpha = ( \bigcup_{\beta<\alpha} \Xi_\beta ) \cup \{ \text{formulas in succedents of this finite branch} \}. \]
Modifications:
Omit step 1.
Enumerate formulas that appear in the active sequent.
-Say, pick the last to have been worked with and it has \( \exists x \varphi(x) \).

**Problem:** The following condition should hold, but does not:
If \( \forall x \varphi(x) \) is in \( \Lambda \), then \( \varphi(t) \) is in \( \Xi \) for all terms \( t \).

## 2 Traditional Proof of Completeness for Uncountable Languages

We'll define sets \( \Lambda_{\alpha}, \Xi_{\alpha} \).

1. There is no finite \( \Gamma_0 \subseteq \Lambda_{\alpha} \) and finite \( \Delta_0 \subseteq \Xi_{\alpha} \) such that \( \Gamma_0 \rightarrow \Delta_0 \) has a proof.
2. \( \Gamma \subseteq \Lambda_{\alpha} \)

Set \( \kappa = \{0, 1, 2, \ldots, \alpha, \ldots \} \)

Where \( A \) is an L-sentence and \( t \) an L-term, enumerate all pairs \( \langle A, t \rangle \):

\[ \langle A_0, t_0 \rangle, \langle A_0, t_1 \rangle, \ldots, \langle A_1, t_0 \rangle, \ldots, \langle A_\alpha, t_\alpha \rangle, \ldots \]

such that for all pairs \( \langle A, t \rangle \) and all \( \beta \in \kappa \) there exists \( \alpha > \beta \) such that \( \langle A, t \rangle = \langle A_\alpha, t_\alpha \rangle \).

For this it is sufficient to assume that each \( \langle A, t \rangle \) appears \( \kappa \)-many times.

Define \( \Lambda, \Xi \) satisfying (1) and (2)
Initially \( \Lambda_{-1} = \Gamma \) and \( \Xi_{-1} = \emptyset \)
Suppose \( \Lambda_\beta, \Xi_\beta \) are defined for all \( \beta < \alpha \)
Let

\[ \Lambda^-_{\alpha} = \bigcup_{\beta < \alpha} \Lambda_{\beta} \]
\[ \Xi^-_{\alpha} = \bigcup_{\beta < \alpha} \Xi_{\beta} \]

Skip step (1). Do step (2) and (3)

**Step (2)**
If \( A_\alpha \in \Lambda^-_{\alpha} \), and \( A_\alpha \) is \( \varphi \land \psi \),
put \( \Lambda_\alpha = \Lambda^-_{\alpha} \cup \{ \varphi, \psi \} \) and \( \Xi_\alpha = \Xi^-_{\alpha} \)

If \( \exists \ \Gamma_0 \subseteq \Lambda_\alpha, \Delta_0 \subseteq \Xi_\alpha \) such that \( \Gamma_0 \rightarrow \Delta_0 \) has a proof,
then \( \Gamma_0 = \Gamma^-_0 \cup \{ \varphi, \psi \} \)

\[
\frac{\Gamma^-_0, \varphi, \psi \rightarrow \Delta_0}{\Gamma_0, \varphi \land \psi \rightarrow \Delta_0}
\]

Where \( \Gamma^-_0, \varphi \land \psi \subseteq \Lambda_{\beta_0} \) for some \( \alpha \beta_0 \) and \( \Delta_0 \subseteq \Xi_{\beta_0} \) for some \( \beta_0 \)

**Step (3)**
If \( A_\alpha \in \Lambda^-_{\alpha} \), and \( A_\alpha \) is \( \varphi \lor \psi \),
put either

\[ \Lambda_\alpha = \Lambda^-_{\alpha} \cup \{ \varphi \} \) and \( \Xi_\alpha = \Xi^-_{\alpha} \)
or
\[ \Lambda_\alpha = \Lambda^-_\alpha \cup \{\psi\} \] and \[ \Xi_\alpha = \Xi^-_\alpha \]
...Whichever satisfies condition (1)

Finally, set \[ \Lambda = \bigcup_{\beta < \kappa} \Lambda_\beta \] and \[ \Xi = \bigcup_{\beta < \kappa} \Xi_\beta \]

So \( \exists \) a model \( M \) such that

1. let \( \sigma(c) = c \)
2. or \( \sigma(c) = [c] \)

Then \( M \models \lambda[c] \) for all \( \lambda \in \Lambda \) and \( M \not\models \{[c] \text{ for all } \} \in \Xi \)

So \( M \) shows \( \Gamma \) is satisfiable.