1 The theory of dense linear order without endpoints

Examples of this theory include \((\mathbb{Q}, <)\) and \((\mathbb{R}, <)\). The language is the set \(\{<, =\}\), and the axioms are as follows:

**Axiom 1** (Linear order). \(\forall x \forall y (x < y \lor y < x \lor y = x)\)

**Axiom 2** (Linear order). \(\forall x (\neg x < x)\)

**Axiom 3** (Transitivity). \(\forall x \forall y \forall z (x < y \land y < z \rightarrow x < z)\)

**Axiom 4** (Without endpoints). \(\forall x (\exists y)(y < x)\)

**Axiom 5** (Without endpoints). \(\forall x (\exists y)(x < y)\)

**Axiom 6** (Dense). \(\forall x \forall y (x < y \rightarrow (\exists z)(x < z \land z < y))\)

We have the following theorem about dense linear order without endpoints:

**Theorem 1.** The theory of dense linear order without endpoints is \(\aleph_0\)-categorical.

Theorem 1 implies the following corollary:

**Corollary 1.** \((\mathbb{Q}, <)\) is the only countable model up to isomorphism.

*Proof of theorem 1.* Let \((M, <^M)\) and \((N, <^N)\) be countable models of \(T\), the theory of dense linear order without endpoints. We’ll use a ”back and forth” argument to construct an isomorphism.

First, enumerate \(|M|\) as \(m_1, m_2, m_3, \ldots\) and enumerate \(|N|\) as \(n_1, n_2, n_3, \ldots\). We want an isomorphism \(f : M \rightarrow N\). We can construct \(f\) in stages \(f_0, f_1, f_2, \ldots\). We want \(f_i\) to have the following properties:

- \(\text{domain}(f_i) \supseteq \{m_1, \ldots, m_i\}\)
- \(\text{range}(f_i) \supseteq \{n_1, \ldots, n_i\}\)
• $f_i \subseteq f_{i+1}$
• $f_i$ is a partial isomorphism.
• $f_i$ is injective.
• $\forall m, m' \in \text{domain}(f_i)$,

$$f_i(m) <^N f_i(m') \iff m <^M m'$$

First, let $f_0 = \emptyset$. Now, to define $f_{i+1}$,

1. If $m \in \text{domain}(f_i)$, then $f_{i+1}(m) = f_i(m)$. Else, find $m, m' \in \text{domain}(f_i)$ such that

$$m <^M m_i <^M m'$$

and not $m'' \in \text{domain}(f_i)$ such that

$$m <^M m'' <^M m'$$

Now consider $f(m) = n$ and $f(m') = n'$. We have $n <^N n'$, so by density, $\exists n^* \text{ such that } n <^N n^* <^N n'$. Set $f_{i+1}(m_i) = n^*$.

Note that $n^* \notin \text{range}(f_i)$ since $f_i^{-1}(n^*)$ would satisfy $m <^M f_i^{-1}(n^*) <^M m'$.

Also note that $\forall m \in \text{domain}(f_i), m'' < m_i \leftrightarrow f_i(m'') < n^*$. Furthermore, if there is no $m <^M m_i$ or $m_i <^M m$, then $m \in \text{domain}(f_i)$.

2. Now assume $n_i \notin \text{range}(f_i)$. Choose $m^* \in |M|$ analogously and set $f_{i+1}(m^*) = n + i$.

Otherwise, $f_i^{-1}(n_i) = f_i^{-1}(n_i)$.

Now let $f = \bigcup_i f_i$. Then we have that

• $f$: 1 − 1 because $f$ is total.
• $f$: onto because $f$ is an isomorphism.

This completes the proof. \qed

Now as a reminder,

**Theorem 2** (Los-Vaught Test). If $T$ has no finite model, and $T$ is $\kappa$-categorical for some $\kappa$ that is greater than the cardinality of the language of $T$, then $T$ is complete.
Corollary 2. The theory of dense linear order without end points is complete.

Corollary 3. \( Th(\text{dense linear order without end points}) \)

\[ = Th(\mathbb{Q}, <) \]
\[ = Th(\mathbb{R}, <) \]

Proof of corollary 3. Let \( \phi \in Th(\mathbb{Q}, <) \), i.e. \( \phi \) is a sentence and \( (\mathbb{Q}, <) \models \phi \)

Let \( T \) be the theory of dense linear order without end points. Either \( T \models \phi \) or \( T \models \neg \phi \) by completion. But if \( T \models \neg \phi \), then \( Th(\mathbb{Q}, <) \models \neg \phi \).

2 Definitions by extension

Let \( T_1 \) be a set of sentences in a language \( L \). Let \( \phi(x_1, ..., x_k) \) be a formula with only \( x_1, ..., x_k \) free in \( \phi \). Augment \( L \) to a bigger language \( L' = L \cup \{p\} \) where \( p \) is a \( k \)-ary predicate symbol.

Form \( T_2 = T_1 \cup \{\forall x_1...\forall x_k(P(x_1, ..., x_k) \leftrightarrow \phi(x_1, ..., x_k))\} \)

Definition 1. Let \( T_1, T_2 \) be a set of sentences in a language \( L_1 \subseteq L_2 \). \( T_2 \) is conservative over \( T_1 \) provided that \( \forall L_1\)-sentences \( A, T_2 \models A \iff T_1 \models A \).

Theorem 3. For \( T_1, T_2 \) as defined above, \( T_2 \) is conservative over \( T_1 \).

Proof of theorem 3. Suppose \( T_2 \models A \). It suffices to show that \( T_1 \models A \). Suppose that this is not the case, and that there exists a model \( M \models T_1 \cup \{\neg A\} \).

Form a new \( M' = T_2 \cup \{\neg A\} \) by creating \( M' \) as the expansion of \( M \) to language \( L' \) with

\[ < m_1, ..., m_k > \in P^{M'} \iff M \models \phi[m_1, ..., m_k] \]

Notation: \( M \models \phi[m_1, ..., m_k] \) iff \( \forall \sigma \) such that \( \sigma(x_i) = m_i \), \( M \models \phi[\sigma] \).

Now it remains to show that

Claim 1. \( M' \models \forall x_1...\forall x_k(P(x_1, ..., x_k) \leftrightarrow \phi(x_1, ..., x_k)) \)

\[ \forall m_1, ..., m_k \in |M'|, < m_1, ..., m_k > \in P^{M'} \]
\[ \iff M' \models \phi[m_1, ..., m_k] \]
\[ M \models \phi[m + 1, ..., m_k] \]

Therefore, \( M' \models T_2 \) and \( M' \models \neg A \).