Math 260A — Mathematical Logic — Scribe Notes
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1 The theory of dense linear order without endpoints

Examples of this theory include $(\mathcal{Q}, <)$ and $(\mathcal{R}, <)$. The language is the set $\{<, =\}$, and the axioms are as follows:

Axiom 1 (Linear order)**.** $\forall x \forall y (x < y \vee y < x \vee y = x)$

Axiom 2 (Linear order)**.** $\forall x (\neg x < x)$

Axiom 3 (Transitivity)**.** $\forall x \forall y \forall z (x < y \wedge y < z \rightarrow x < z)$

Axiom 4 (Without endpoints)**.** $\forall x (\exists y)(y < x)$

Axiom 5 (Without endpoints)**.** $\forall x (\exists y)(x < y)$

Axiom 6 (Dense)**.** $\forall x \forall y (x < y \rightarrow (\exists z)(x < z \wedge z < y))$

We have the following theorem about dense linear order without endpoints:

Theorem 1. *The theory of dense linear order without endpoints is $\aleph_0$-categorical.*

Theorem 1 implies the following corollary:

Corollary 1. $(\mathcal{Q}, <)$ *is the only countable model up to isomorphism.*

Proof of theorem 1. Let $(\mathcal{M}, <^{\mathcal{M}})$ and $(\mathcal{N}, <^{\mathcal{N}})$ be countable models of T, the theory of dense linear order without endpoints. We'll use a "back and forth" argument to construct an isomorphism.

First, enumerate $|\mathcal{M}|$ as $m_1, m_2, m_3, ...$ and enumerate $|\mathcal{N}|$ as $n_1, n_2, n_3, ...$ We want an isomorphism $f : \mathcal{M} \rightarrow \mathcal{N}$. We can construct f in stages $f_0, f_1, f_2, ...$ We want f_i to have the following properties:

- $domain(f_i) \supseteq \{m_1, ..., m_i\}$
- $range(f_i) \supseteq \{n_1, ..., n_i\}$

- $f_i \subseteq f_{i+1}$
- f_i is a partial isomorphism.
- f_i is injective.
- $\forall m, m' \in domain(f_i)$,

$$f_i(m) <^{\mathcal{N}} f_i(m') \Leftrightarrow m <^{\mathcal{M}} m'$$

First, let $f_0 = \emptyset$. Now, to define f_{i+1},

1. If $m \in domain(f_i)$, then $f_{i+1}(m) = f_i(m)$. Else, find $m, m' \in domain(f_i)$ such that

   $$m <^{\mathcal{M}} m_i <^{\mathcal{M}} m'$$

   and not $m'' \in domain(f_i)$ such that

   $$m <^{\mathcal{M}} m'' <^{\mathcal{M}} m'$$

   Now consider $f(m) = n$ and $f(m') = n'$. We have $n <^{\mathcal{N}} n'$, so by density, $\exists n^*$ such that $n <^{\mathcal{N}} n^* <^{\mathcal{N}} n'$. Set $f_{i+1}(m_i) = n^*$.

   Note that $n^* \notin range(f_i)$ since $f_i^{-1}(n^*)$ would satisfy $m <^{\mathcal{M}} f_i^{-1}(n^*) <^{\mathcal{M}} m'$.

   Also note that $\forall m \in domain(f_i), m'' < m_i \leftrightarrow f_i(m'') < n^*$. Furthermore, if there is no $m <^{\mathcal{M}} m_i$ or $m_i <^{\mathcal{M}} m$, then $m \in domain(f_i)$.

2. Now assume $n_i \notin range(f_i)$. Choose $m^* \in |\mathcal{M}|$ analogously and set $f_{i+1}(m^*) = n + i$.

   Otherwise, $f_{i+1}^{-1}(n_i) = f_i^{-1}(n_i)$.

Now let $f = \bigcup_i f_i$. Then we have that

- f: $1 - 1$ because f is total.
- f: onto because f is an isomorphism.

This completes the proof. □

Now as a reminder,

Theorem 2 (Łos-Vaught Test)**.** *If T has no finite model, and T is κ-categorical for some κ that is greater than the cardinality of the language of T, then T is complete.*

Corollary 2. *The theory of dense linear order without end points is complete.*

Corollary 3. $Th(\textit{dense linear order without end points})$

$$= Th(\mathcal{Q}, <)$$
$$= Th(\mathcal{R}, <)$$

Proof of corollary 3. Let $\phi \in Th(\mathcal{Q}, <)$, i.e. ϕ is a sentence and

$$(\mathcal{Q}, <) \models \phi$$

Let T be the theory of dense linear order without end points. Either $T \models \phi$ or $T \models \neg\phi$ by completion. But if $T \models \neg\phi$, then $Th(\mathcal{Q}, <) \models \neg\phi$. □

2 Definitions by extension

Let T_1 be a set of sentences in a language L. Let $\phi(x_1, ..., x_k)$ be a formula with only $x_1, ..., x_k$ free in ϕ. Augment L to a bigger language $L' = L \cup \{p\}$ where p is a k-ary predicate symbol.

Form $T_2 = T_1 \cup \{\forall x_1 ... \forall x_k (P(x_1, ..., x_k) \leftrightarrow \phi(x_1, ..., x_k))\}$

Definition 1. Let T_1, T_2 be a set of sentences in a language $L_1 \subseteq L_2$. T_2 is conservative over T_1 provided that $\forall L_1$-sentences A, $T_2 \models A \Leftrightarrow T_1 \models A$.

Theorem 3. *For T_1, T_2 as defined above, T_2 is conservative over T_1.*

Proof of theorem 3. Suppose $T_2 \models A$. It suffices to show that $T_1 \models A$. Suppose that this is not the case, and that there exists a model $\mathcal{M} \models T_1 \cup \{\neg A\}$. Form a new $\mathcal{M}' \models T_2 \cup \{\neg A\}$ by creating $\mathcal{M}'$ as the expansion of $\mathcal{M}$ to language L' with

$$< m_1, ..., m_k > \in P^{\mathcal{M}'} \Leftrightarrow \mathcal{M} \models \phi[m_1, ..., m_k]$$

Notation: $\mathcal{M} \models \phi[m_1, ..., m_k]$ iff $\forall \sigma$ such that $\sigma(x_i) = m_i$, $\mathcal{M} \models \phi[\sigma]$.

Now it remains to show that

Claim 1. $\mathcal{M}' \models \forall x_1 ... \forall x_k (P(x_1, ..., x_k) \leftrightarrow \phi(x_1, ..., x_k))$

$$\forall m_1, ..., m_k \in |\mathcal{M}'|, < m_1, ..., m_k > \in P^{\mathcal{M}'}$$
$$\Leftrightarrow \mathcal{M}' \models \phi[m_1, ..., m_k]$$
$$\mathcal{M} \models \phi[m + 1, ..., m_k]$$

Therefore, $\mathcal{M}' \models T_2$ and $\mathcal{M}' \models \neg A$. □