1 Robinson resolution refutation

Let $\Gamma$ be a set of clauses of first order literals — the terms have the form $P(t_1, \ldots, t_k)$ or $\neg P(t_1, \ldots, t_k)$ for terms $t_1, \ldots, t_k$ and $k$-ary function $P$. Without loss of generality, we will assume the clauses in $\Gamma$ use distinct variables (though terms within a clause cannot be assumed distinct).

Throughout these notes, we will assume the language $L$ contains at least one constant symbol.

**Definition** A ground resolution refutation of $\Gamma$ is a sequence of clauses $C_1, C_2, \ldots, C_k = \emptyset$ where each $C_i$ is either a ground instance of a clause in $\Gamma$, or is inferred by a resolution inference from two previous clauses $C_j$ and $C_{\ell}$.

**Definition** A Robinson resolution refutation of $\Gamma$ is a sequence of clauses $C_1, C_2, \ldots, C_k = \emptyset$ where each $C_i$ is either a relabeling\(^1\) of a clause in $\Gamma$, or is obtained by a Robinson resolution inference from two previous clauses $C_j$ and $C_{\ell}$.

To define a Robinson resolution inference, take two sets of clauses $A$ and $B$, and nonempty subsets $A' \subset A$, $B' \subset B$, where $A'$ contains only positive clauses, and $B'$ has only negative clauses. Let

$$F = \{ \varphi \mid \varphi \in A' \} \cup \{ \neg \varphi \mid \varphi \in B' \}.$$  

Choose an mgu $\sigma$ unifying $F$, so that $\varphi\sigma = P(t_1, \ldots, t_k)$ for every $\varphi \in F$. If such a $\sigma$ exists, then we make this resolution inference:

$$\frac{A\sigma}{C} \frac{B\sigma}{C} \text{ Resolution}$$

where $C = (A \setminus A')\sigma \cup (B \setminus B')\sigma$.

\(^1\)Traditionally a Robinson resolution does not allow for relabeling the variables in a clause. We allow it here as it does not add any power, but removes some technical concerns from the upcoming proof.
For $C$ determined from $A$ and $B$ by such an inference, we have

$$\frac{A}{C} \frac{B}{C}$$

Robinson resolution

The selection of $A'$ and $B'$ is called factoring.

Note At first glance, this inference rule may seem needlessly complex. Why not do resolution on individual terms of the clause? There’s a good reason: such inferences are not complete. Here is a simple example where things go wrong.

$$\Gamma = \{ \{P(x),P(y)\},\{\neg P(u),\neg P(v)\}\}.$$  
$\Gamma$ corresponds to the sentence

$$(\forall x \forall y P(x) \lor P(y)) \land (\forall x \forall y \neg P(u) \lor \neg P(v)).$$

This is clearly unsatisfiable,

However, the only inference possible from these clauses (up to variable names) is to resolve $P(x)$ against $\neg P(u)$ (after appropriate unification), which leaves us with the resolvent $\{P(y),\neg P(v)\}$, which corresponds to the sentence

$$\forall y \forall v P(x) \lor \neg P(y),$$

which is a tautology, and thus not any help.

1.1 Relation to ground resolution refutation

Theorem If $\Gamma$ has a ground resolution refutation, then $\Gamma$ has a Robinson resolution refutation.

Note This theorem saves us from choosing terms for the ground instances, instead requiring a good factoring strategy.

Proof Let $C_1,\ldots,C_k = \emptyset$ be a ground resolution refutation. Without loss of generality, we assume that $C_i \neq C_j$

We will find a Robinson resolution refutation $D_1,\ldots,D_k$ on distinct variables, and substitutions $\sigma_1,\ldots,\sigma_k$ such that $C_i = D_i \sigma_i$. In particular, $D_k = \emptyset$.

We will show that, if the above property holds for the initial sequence $C_1,\ldots,C_{i-1}$, then it also holds for $C_1,\ldots,C_i$.

Case 1 $C_i$ is a ground instance of $C \in \Gamma$. Let $D_i$ be an instance of $C$ with new variables (not yet seen), so $C_i$ is a substitution instance of $D_i$. Pick such a substitution $\sigma_i$. 

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Case 2  \( C_i \) is the resolvent of \( C_j = D_j \sigma_j \) and \( C_\ell = D_\ell \sigma_\ell \), with respect to \( P(t) \). Select \( D'_j = \{ \varphi \in D_j \mid \varphi \sigma = P(t) \} \), and \( D'_\ell = \{ \varphi \in D_\ell \mid \varphi \sigma = \neg P(t) \} \). Let

\[ F = \{ \varphi \mid \varphi \in D'_j \} \cup \{ \varphi \mid \neg \varphi \in D'_\ell \}. \]

Since the \( D \)'s are chosen to have distinct variables, the domains of \( \sigma_j, \sigma_\ell \) are disjoint.

By construction, \( \sigma_j \cup \sigma_\ell \) unifies \( F \), so \( F \) must have an mgu — call it \( \tau \) — so that \( \exists \pi, \tau \pi = \sigma_j \cup \sigma_\ell \). Choose such a \( \tau \) which sends all variables in \( C_j \) and \( C_\ell \) to a new set of unused variables.

Let \( D_i = \text{Robinson resolvent} = (D_j \setminus D'_j) \tau \cup (D_\ell \setminus D'_\ell) \tau \).

Claim  \( C_i = D_i \pi \).

Proof

\[ \psi \in C_i \iff \psi \in (C_j \setminus \{ P(t) \}) \cup (C_\ell \setminus \{ \neg P(t) \}) \]

\[ \iff \exists \psi' \in D_j \setminus D'_j, \psi = \psi' \sigma_j, \text{ or } \exists \psi' \in D_\ell \setminus D'_\ell, \psi = \psi' \sigma_\ell \]

\[ \iff \exists \psi' \in (D_j \setminus D'_j) \cup (D_\ell \setminus D'_\ell), \psi = \psi'(\sigma_j \cup \sigma_\ell) = \psi' \tau \pi \]

\[ \iff \exists \psi' \in D_i, \psi = \psi' \pi \]

which was the goal. Take \( \sigma_i = \pi \), so \( C_i = D_i \sigma_i \), completing the proof.