I will be denoting $S(x)$, the successor of $x$ as $Sx$

**Last Time**

Recall in the last lecture we defined the theory $Q$:

- Usual FO symbols (including propositional connectives, quantifiers, equality)
- Non-logical symbols ($0, S, +, ·$)
- Axioms:
  1. $\forall x \,(Sx \neq 0)$
  2. $\forall x \forall y \,((Sx = Sy) \rightarrow (x = y))$
  3. $\forall x \,(x \neq 0 \rightarrow \exists y \,(Sy = x))$
  4. $\forall x \,(x + 0 = x)$
  5. $\forall x \forall y \,((x + Sy) = S(x + y))$
  6. $\forall x \,(x \cdot 0 = 0)$
  7. $\forall x \forall y \,((x \cdot Sy) = (x \cdot y) + x)$

**Definition 1.** $Q_{\leq}$: The conservative extension of $Q$ that includes the inequality symbol $\leq$ by adding the axiom $x \leq y \leftrightarrow \exists z \,(x + z = y)$

**Definition 2.** A theory is said to be *bounded* if it is axiomatizable with a set of bounded formulas. We want to be able to treat bounded quantifiers separately from regular quantifiers.

$Q, Q_{\leq}$ are induction-free fragments of arithmetic. The axioms of $Q, Q_{\leq}$ do not imply many elementary facts about addition and multiplication, such as commutativity and associativity. We want a language stronger than $Q$.  

---

1Based on handwritten class notes by Tanya Hall
1 Induction Axioms

Let $A(x)$ be a formula. Induction axiom for $A$ is

$$A(0) \land (\forall x (A(x) \rightarrow A(Sx)) \rightarrow \forall x A(x))$$

$A(x)$ can have other free variables (parameters). The axiom for $A(x, \vec{y})$ is

$$A(0, \vec{y}) \land (\forall x (A(x, \vec{y}) \rightarrow A(Sx, \vec{y})) \rightarrow \forall x A(x, \vec{y}))$$

**Definition 3.** The theory of Peano Arithmetic, $PA$, is the theory $Q_{\leq}$ plus induction for all first-order formulas.

2 Minimization Axioms

The following are two equivalent statements of the minimization axioms:

$$\exists x A(x) \rightarrow \exists x (A(x) \land \forall y (y < x \rightarrow \neg A(y)))$$

$$\exists x A(x) \rightarrow \exists x (A(x) \land \neg \exists y (y < x \land A(y)))$$

Note that while $<$ is not technically in the language, we can use $y < x$ to abbreviate $y \leq x \land y \neq x$.

The Minimization Axioms are often used as an equivalence to Complete Induction.

3 Complete Induction

$$\forall x [\forall y (y < x \rightarrow \neg A(y)) \rightarrow \neg A(x)] \rightarrow \forall x \neg A(x)$$

If we take $B$ to be $\neg A$ and push negations, then this is equivalent to minimization on $\neg B$. We will now show that induction on $\neg A$ is equivalent to minimization on $A$

$$\forall x [\forall y (y < x \rightarrow \neg A(y)) \rightarrow \neg A(x)] \rightarrow \forall x \neg A(x)$$

$$\neg \forall x \neg A(x) \rightarrow \neg \exists x [\forall y (y < x \rightarrow \neg A(y)) \rightarrow \neg A(x)]$$

$$\exists x A(x) \rightarrow \exists x [\neg A(x) \land \forall y (y < x \rightarrow \neg A(y))]$$

$$\exists x A(x) \rightarrow \exists x [\neg A(x) \land \forall y (y < x \rightarrow \neg A(y))]$$

On the face of it, complete induction is weaker than ordinary induction because you have to assume more; the antecedent is stronger.
4 Power of Induction

What is induction good for? Unlike in $Q/Q^<_r$, with induction we get basic facts about addition and multiplication. For example, PA implies commutativity of addition:

**Claim 1.** $PA \vdash \forall x \forall y (x + y = y + x)$

*Proof.* by induction on $x$. Let $A(x, y)$ be $x + y = y + x$.

We will use the induction axiom $A(0, y) \land (\forall x \left( A(x, y) \rightarrow A(Sx, y) \right) \rightarrow \forall x A(x, y))$

So, we need to show

1. $PA \vdash 0 + y = y + 0$
2. $PA \vdash (x + y = y + x) \rightarrow (Sx + y = y + Sx)$

(1) $PA \vdash 0 + y = y + 0$

Since $y + 0 = y$ from an axiom, it is sufficient to show $PA \vdash 0 + y = y$

(1*) $PA \vdash 0 + y = y$

*Proof.* by induction on $y$. Let $B(y)$ be $0 + y = 0$.

Using Induction Axiom $B(0) \land (\forall y \left( B(y) \rightarrow B(Sy) \right) \rightarrow \forall y B(y))$

We need to show

(a) $PA \vdash 0 + 0 = 0$
(b) $PA \vdash (0 + y = y) \rightarrow (0 + Sy = Sy)$

(a) $PA \vdash 0 + 0 = 0$

*Proof.* $0 + 0 = 0$ (by axiom) \hfill \Box

(b) $PA \vdash (0 + y = y) \rightarrow (0 + Sy = Sy)$

1. $0 + y = y$ (by hypothesis)

*Proof.* $0 + Sy = S(0 + y)$ (by axiom) \hfill \Box

2. $0 + Sy = Sy$ (by 1,2)

Thus from (a), (b) with induction $PA \vdash 0 + y = y$ (concluding 1*)

Thus, $PA \vdash 0 + y = y + 0$ (concluding 1)
(2*) \( PA \vdash Sx + y = S(x + y) \)

\[ \text{Proof.} \text{ by induction on } y. \text{ Let } C(x, y) \text{ be } Sx + y = S(x + y). \]

Using Induction Axiom \( C(x, 0) \land (\forall y \, (C(x, y) \rightarrow C(x, Sy)) \rightarrow \forall y C(x, y)) \)

We need to show

\[ (c) \ PA \vdash Sx + 0 = S(x + 0) \]

\[ (d) \ PA \vdash (Sx + y = S(x + y)) \rightarrow (Sx + Sy = S(x + Sy)) \]

\begin{align*}
(c) & \quad \text{PA} \vdash Sx + 0 = S(x + 0) \\
\text{Proof.} & \\
1. & \quad Sx + 0 = Sx \quad \text{(by axiom)} \\
2. & \quad x + 0 = x \quad \text{(by axiom)} \\
3. & \quad Sx + 0 = S(x + 0) \quad \text{(by 1,2)}
\end{align*}

\begin{align*}
(d) & \quad \text{PA} \vdash (Sx + y = S(x + y)) \rightarrow (Sx + Sy = S(x + Sy)) \\
\text{Proof.} & \\
1. & \quad Sx + y = S(x + y) \quad \text{(by hypothesis)} \\
2. & \quad x + Sy = S(x + y) \quad \text{(by axiom)} \\
3. & \quad S(x + Sy) = S(S(x + y)) \quad \text{(by axiom)} \\
4. & \quad Sx + Sy = S(Sx + y) \quad \text{(by axiom)} \\
5. & \quad Sx + Sy = S(S(x + y)) \quad \text{(by 1, 4)} \\
6. & \quad Sx + Sy = S(x + Sy) \quad \text{(by 3, 5)}
\end{align*}

Thus from \((c), (d)\) and induction axiom \( \text{PA} \vdash Sx + y = S(x + y) \) (concluding \(2^*\))

\begin{align*}
1. & \quad y + Sx = S(y + x) \quad \text{(by axiom)} \\
2. & \quad x + y = y + x \quad \text{(by hypothesis)} \\
3. & \quad y + Sx = S(x + y) \quad \text{(by 1, 2)} \\
4. & \quad Sx + y = y + Sx \quad \text{(by } 2^*, 3) \\
\end{align*}

Thus \( \text{PA} \vdash (x + y = y + x) \rightarrow (Sx + y = y + Sx) \) (concluding \(2\))

Thus from (1) and (2) and induction, \( \text{PA} \vdash (x + y = y + x) \)
5 Some things PA can prove

a) Addition is commutative: \( \forall x \forall y (x + y = y + x) \)
b) Addition is associative: \( \forall x \forall y \forall z ((x + y) + z = x + (y + z)) \)
c) Multiplication is commutative: \( \forall x \forall y (x \cdot y = y \cdot x) \)
d) Distributive law: \( \forall x \forall y \forall z ((x + y) \cdot z = x \cdot z + y \cdot z) \)
e) Multiplication is associative: \( \forall x \forall y \forall z ((x \cdot y) \cdot z = x \cdot (y \cdot z)) \)
f) Cancellation laws for addition: \( \forall x \forall y \forall z (x + z = y + z \leftrightarrow x = y) \)
and \( \forall x \forall y \forall z (x + z \leq y + z \leftrightarrow x \leq y) \)
g) Discreteness of \( \leq \): \( \forall x \forall y (x \leq Sy \rightarrow x \leq y \lor x = Sy) \)
h) Transitivity of \( \leq \): \( \forall x \forall y \forall z (x \leq y \land y \leq z \rightarrow x \leq z) \)
i) Anti-idempotency laws: \( \forall x \forall y (x + y = 0 \rightarrow x = 0 \land y = 0) \) and
\( \forall x \forall y (x \cdot y = 0 \rightarrow x = 0 \lor y = 0) \)
j) Reflexivity, trichotomy and antisymmetry of \( \leq \): \( \forall x (x \leq x), \forall x \forall y (x \leq y \lor y \leq x), \forall x \forall y \forall z (x \leq y \land y \leq x \rightarrow x = y) \)
k) Cancellation laws for multiplication: \( \forall x \forall y \forall z (z \neq 0 \land x \cdot z = y \cdot z \rightarrow x = y) \)
and \( \forall x \forall y \forall z (z \neq 0 \land x \cdot z \leq y \cdot z \rightarrow x \leq y) \)

6 Prove \( Q \vdash \forall x \neg (x < 0) \)

Proof. Suppose \( x < 0 \). This means \( x \leq 0 \land x \neq 0 \). \( x \leq 0 \) means \( \exists z (x + z = 0) \).
By \( Q \) axiom, either \( z = 0 \) or \( \exists z' \) such that \( S z' = z \).
If \( z = 0 \), then \( 0 = x + z = x + 0 = x \) which contradicts the fact \( x \neq 0 \)
If \( z = S z' \), then \( 0 = x + S z' = S (x + z) \) which contradicts the axiom \( \forall x (S x \neq 0) \)
7 Complete Induction Axioms redux

We will now show $PA \vdash \forall x [\forall y (y < x \rightarrow A(y)) \rightarrow A(x)] \rightarrow \forall x A(x)$

We are going to use induction on the statement $B(x) = \forall y (y < x \rightarrow A(y))$

Proof. Assume the hypothesis $\forall x [\forall y (y < x \rightarrow A(y)) \rightarrow A(x)]$.

Base Case $B(0)$ is $\forall y (y < 0 \rightarrow A(y))$; so $Q \vdash B(0)$

Induction Step Assume $B(x)$. We want to show $B(Sx)$

So we assume $\forall y (y < x \rightarrow B(y))$, and want to prove that $(y < Sx \rightarrow B(y))$.

Assume $y < Sx$. By discreteness, we know $y \leq x$. This in turn means $y < x \lor y = x$.

If $y < x$, then $B(y)$ holds by our inductive hypothesis that $\forall y (y < x \rightarrow B(y))$.

If $y = x$, then $B(y)$ holds from the hypothesis $\forall x [\forall y (y < x \rightarrow A(y)) \rightarrow A(x)]$

Thus by induction, $PA \vdash \forall y B(y)$. In particular, let $x$ be arbitrary $B(Sx)$. Since $x < Sx$, $A(x)$ holds.

8 $1 + 1 = 2$

Proof. Define 1 := $S0$, and define 2 := $SS0$

$S0 + S0 = S (S0 + 0) = SS (0 + 0) = SS0$