1 Quantifier Complexity and Bounded Quantifiers

So far we have used ordinary quantifiers $\forall$ and $\exists$. In order to study quantifier complexity, we now introduce bounded versions, defined here:

$$(\forall y \leq t)A(y) \leftrightarrow (\forall y)(y \leq t \rightarrow A(y))$$

$$(\exists y \leq t)A(y) \leftrightarrow (\exists y)(y \leq t \land A(y))$$

where $t$ is a term not involving $y$.

Define a formula to be $\Delta_0$ if all of its quantifiers are bounded.

We further define a sequence of classes of formulas.

- A $\Sigma_1$ formula has the form $(\exists y_1)\ldots(\exists y_k)\varphi(x, \vec{y})$, where $\varphi$ is $\Delta_0$.
- A $\Pi_1$ formula has the form $(\forall y_1)\ldots(\forall y_k)\varphi(x, \vec{y})$, where $\varphi$ is $\Delta_0$.
- A $\Sigma_2$ formula has the form $(\exists y)(\forall z)\varphi(x, \vec{y}, z)$, where $\varphi$ is $\Delta_0$. Equivalently, it has the form $(\exists y)\psi(x, \vec{y})$, where $\psi$ is $\Pi_1$.
- Inductively, a formula is $\Sigma_n$ if it has the form $\exists y\varphi(x, \vec{y})$, where $\varphi$ is $\Pi_{n-1}$.
- $\Pi_n$ is defined dually.

Note In each of the above, we may take any of the quantifier blocks to be empty so that, for example, $\Sigma_n \subseteq \Sigma_{n+1}$.

We now consider restricted induction axioms. If $\Phi$ is a class of formulas (such as $\Delta_0$ or $\Sigma_3$), the $\Phi$ induction axioms are

$$\{A(0) \rightarrow (\forall x)(A(x) \rightarrow A(Sx)) \rightarrow (\forall x)A(x) : A \in \Phi\}.$$ 

Denote by $I\Phi$ the axiom system $Q_{\leq} + \Phi$-induction axioms\(^1\) For example, $I\Delta_0$ allows induction on all $\Delta_0$ formulas. Last time we showed that $I\Delta_0$ proves $x + y = y + x$.

\(^1\)It is possible to redefine the axioms of $Q_{\leq}$ to use only bounded quantifiers.
We define Peano Arithmetic = \( PA = \bigcup I \Sigma_n = \bigcup I \Pi_n \).

More generally, we define classes \( \Sigma_n^+ \) and \( \Pi_n^+ \). \( \Sigma_2 \), for example, includes formulas of the form \( (\forall u \leq t)\exists y \forall v \varphi(u, x, y, z) \), where \( \varphi \) is \( \Delta_0 \). Simply put, you get a \( \Sigma_n^+ \) formula by taking any \( \Sigma_n \) formula and inserting bounded quantifiers wherever you like — including inside of a quantifier block.

**Collection property / replacement property**

The following is valid in \( \mathbb{N} \):

\[
(\forall y \leq t)(\exists z)\varphi(y, z) \Rightarrow (\exists u)(\forall y \leq t)(\exists z \leq u)\varphi(y, z).
\]

(1)

This serves to put a uniform bound on the \( z \)-values, which is possible since there are only finitely many \( y \) values being considered.

If \( \varphi \) is in \( \Sigma_n \) (for example), then 1 is called a \( \Sigma_n \)-replacement axiom.

**Theorem** Any \( \Sigma_n^+ \) formula is equivalent to a \( \Sigma_n \) formula, and any \( \Pi_n^+ \) formula is equivalent to a \( \Pi_n \) formula.

We prove the statement for \( \Sigma_n^+ \), and \( \Pi_n^+ \) follows dually.

Since the converse to 1 is trivial, we will instead show both directions.

We work by induction on \( n \).

We take the inverse of the axiom, so we’ll instead show:

\[
(\exists y \leq t)\forall z \varphi(y, z) \Leftrightarrow \forall u \exists y \leq u \forall z \leq u \varphi(y, z),
\]

where \( \psi = \neg \phi \).

Thus it is enough to show that, if \( \chi \in \Sigma_n \), then so are \( (\forall y \leq t)\chi \) and \( (\exists y \leq t)\chi \).

Since \( \chi \in \Sigma_n \), it has the form \( \exists z_1 \ldots \exists z_k \varphi(y, z) \). Thus we have

\[
(\forall y \leq t) \quad \Leftrightarrow \quad (\forall y \leq t)\exists z_1 \ldots \exists z_k \varphi(y, z) \\
\Leftrightarrow \quad (\exists u)(\forall y \leq t)(\exists z_1 \leq u)\exists z_2 \ldots \exists z_k \varphi(y, z) \\
\Leftrightarrow \quad (\exists u)(\forall y \leq t)\exists z_2 \ldots \exists z_k (\exists z_1 \leq u)\varphi(y, z) \\
\Leftrightarrow [\text{repeat-1 times}] \\
\Leftrightarrow \quad (\exists u')(\forall y \leq t)(\exists z_1 \leq u') \ldots (\exists z_k \leq u') \varphi(y, z).
\]

Note that the last portion of this formula, \( (\exists z_1 \leq u') \ldots (\exists z_k \leq u') \varphi(y, z) \), is a \( \Pi_{n-1}^+ \) formula, so the induction hypothesis gives us an equivalent \( \Pi_{n-1} \) formula \( \psi'(y, u) \). This formula can now absorb the \( (\forall y \leq t) \). Adding on the \( (\exists u') \) on the front leaves us with a \( \Sigma_n \) formula, as desired.

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\( ^2 \)You may find yourself asking: “Are we allowed to do induction here?” Remember: we are doing induction ourselves, not in a restricted proof system.
Extending languages

We often use $I\Delta_0$ as our base theory. This can prove statements like $x + y = y + x$, which we proved earlier using quantifier-free induction.

Our language is only $\{0, S, +, \cdot, \leq\}$, but we will also want function symbols. We extend by conservative definitions, for example

$$z|x \leftrightarrow \exists u(z \cdot u = x)$$

and

$$Prime(x) \leftrightarrow x \neq 1 \land (\forall z)(z|x \rightarrow z = 1 \lor z = x).$$

These definitions are fine, but we should be mindful of unbounded quantifiers. In both cases, we can (and should) replace them with bounded quantifiers — both $u$ and $z$ may be bounded by $x$ without changing the meaning.

**Definition** A predicate $R(x_1, \ldots, x_k) \subseteq \mathbb{N}^k$ is $\Delta_0$ if there is a $\Delta_0$ formula $\phi(x)$ so that

$$\mathbb{N} \models \forall x(R(x) \leftrightarrow \phi(x)).$$

**Definition** Let $T$ be a theory. Let $R$ be as above. Then $T(R)$ is the theory $T$ in the language of $T$ plus symbol $R$, whose axioms are the axioms of $T$, along with the axiom $Defn_R := \forall \bar{x}(R(\bar{x}) \leftrightarrow \phi(\bar{x}))$.

**Theorem**

(a) $T(R)$ is a conservative extension of $T$.

(b) Let $T$ be a theory from $I\Delta_0, I\Sigma_m, I\Pi_n$. Any bounded (ie $\Delta_0$) formula $\psi$ of $T(R)$ is $T(R)$-provably equivalent to a bounded formula $\chi$ in the language of $T$.

**Proof**

We showed (a) last quarter.

For (b), find $\chi$ by replacing each instance of $R(\bar{t})$ in $\psi$ by $\phi(\bar{t})$. This maintains the quantifier complexity, since $\phi$ is $\Delta_0$.

Note: if $\psi$ is $\Sigma_n$, so is $\chi$, independent of the theory we’re working in.

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