1 Σ₁-defined Functions

Our goal has been to introduce Δ₀ defined relation symbols $R$ and Σ₁-defined function symbols $f$ to be added to $IΔ₀$, and which are usable in induction. So we need Σ₁-defined functions now.

Recall that $f$ is Σ₁-defined means that there is a Δ₀-formula $ϕ$ (or a Σ₁-formula $ϕ$) such that:

- $T ⊃ IΔ₀$
- $T ⊢ ∀\vec{x}∃!y(ϕ(\vec{x}, y))$
- $T(f)$ is $T ∪ \{∀\vec{x}(f(\vec{x})) = y ↔ ϕ(\vec{x}, y)\}$.

Note that $T(f)$ is conservative over $T$.

Claim. For $T$ a bounded theory, any Δ₀($f$) formula is $T(f)$-provably-equivalent to a Δ₀ formula.

First let’s make a preliminary remark. By Parikh’s Theorem, for $T ⊃ IΔ₀$ and $T$ bounded, if $T ⊨ ∀\vec{x}∃!y(ϕ(\vec{x}, y))$, then there is a term $q$ such that

$$T ⊨ ∀\vec{x}∃y ≤ q(ϕ(\vec{x}, y)) .$$

And if $ϕ$ is $∃z₁ \ldots ∃zₖψ(\vec{x}, y, \vec{z})$ then

$$T ⊨ ∀\vec{x}∃y ≤ q∃z₁ ≤ r₁ \ldots ∃zₖ ≤ rₖ(ψ(\vec{x}, y, \vec{z})) ,$$

again by Parikh’s Theorem. So we have that Σ₁ definable implies Δ₀ definable.

Now back to the above claim. Let $ψ$ be a Δ₀($f$) formula. We want an equivalent $ψ^*$ such that $ψ^*$ is Δ₀ and $T(f) ⊨ ψ ↔ ψ^*$. Here’s our idea: We will remove occurrences of $f$ one at a time.

So find an atomic formula containing an occurrence of $f$ of form $s = t$ or $s ≤ t$. That is,

$$s(\ldots f(r₁, \ldots, rₖ)\ldots) = t$$
where there are no $f$’s inside the terms $r_i$. This is equivalent to (in $T(f)$)

$$(\exists y \leq q(r_1, \ldots r_k))(\varphi(\vec{x}, y) \land s(y \ldots) = t)$$

and this is clearly $\Delta_0$.

Alternatively, we could consider

$$(\forall y \leq q(\vec{r}))((\varphi(\vec{x}, y) \rightarrow s(y \ldots) = t)$$

which is also $\Delta_0$. In either case, we have shown the above claim.

Here’s a more general theorem, which we will present without complete proof (since the proof is so similar to the discussion above).

**Theorem 1.** If $f(\vec{x}) = y$ is $\Sigma_1$-defined and if $T \supset B\Sigma_1$, then any $\Sigma_i(f)$ formula (or $\Pi_i(f)$ formula) is $T(f)$ provably equivalent to a $\Sigma_i$ (respectively $\Pi_i$) formula.

The proof of this theorem works exactly in an analogous way as above, except without the bounds on the quantifiers.

## 2 Some Bootstrapping

We have already introduced some things with $\Delta_0$ definitions:

- Restricted subtraction: $x \div y$
- $x$ is prime
- $x$ divides $y$: $x|y \leftrightarrow \exists z \leq y(x \cdot z = y)$

**Claim.** $I\Delta_0 \vdash (x$ is prime $\land x|a \cdot b \rightarrow x|a \lor x|b$).

But we’d like to have

$I\Delta_0 \vdash \forall x (x$ has a unique prime factorization $).$

And in $I\Delta_0$ how can we even say that $x$ has a unique prime factorization? We want something along the lines of

$$\forall \vec{x} \exists p_1, p_2, \ldots p_k (\forall i, p_i \text{ is prime and } x = p_1 \cdot \ldots \cdot p_k) .$$

Or more specifically:

$$\forall \vec{x} \exists (p_1, \ldots p_k)(\forall i p_i \text{ is prime and } x = \Pi_i^k p_i) .$$

So we are almost there, we just need to talk a little bit about sequence coding. But here is a problem:
Theorem 2. $I\Delta_0$ does not $\Delta_0$-define the exponentiation function.

Proof. Suppose it did. So

$$\varphi(x, y) \leftrightarrow x = 2^y$$

and

$$I\Delta_0 \vdash \forall y \exists x \varphi(x, y).$$

Then by Parikh’s Theorem,

$$I\Delta_0 \vdash \forall y \exists x \leq s(y) \varphi(x, y)$$

for some term $s$. And $s$ is made up of 0, $S$, $+$, $\cdot$, so $s$ is a polynomial.

So $s(y) \leq y^l + l$ for some $l \in \mathbb{N}$. So $\mathbb{N} \models \forall y \exists x \leq y^l + l(x = 2^y)$, which is clearly not true since $\forall y(2^y < y^l + l)$ is not true. \hfill $\square$

And this is an issue for sequence coding since the Gödel method used prime powers.

Thus we have that any $\Delta_0$ definable function of $I\Delta_0$ is bounded by a polynomial $s(y) \leq y^l + l$. So we have completely characterized the growth rates of function provably definable in $I\Delta_0$. (Note that this is what computer scientists call “linear growth rate functions.”)

3 Some More Bootstrapping

Let’s define some more things:

• Predecessor: $P(x) = x \div 1 = x \div S(0)$

• Integer division: $x, y \mapsto \lfloor x/y \rfloor$ with

$$\varphi(x, y, z) \leftrightarrow (y \cdot z \leq x \land y \cdot Sz > x) \lor (y = 0 \land z = 0).$$

• $x \mod y = x - y \cdot \lfloor x/y \rfloor$.

• $\lfloor \sqrt{x} \rfloor = z \iff z \cdot z \leq x \land Sz \cdot Sz > x$.

• $x$ is prime (we already did)

• $x$ is a prime power $\iff \exists p \leq x(p$ is prime and $\forall z(z \rightarrow z = 1 \lor p|z)$.

• $x$ is a power of 2 (as in the previous bullet)

• $(x$ is a power of 4$) \leftrightarrow (x$ is a power of 2$ \land (x \mod 3 = 1))$. 

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