Math 260AB - Introduction to Mathematical Logic
Winter and Spring 2008
Instructor: Sam Buss

Homework I. Due: Monday, January 23.

General instructions for all homeworks: Proofs are not required unless the problem asks for a proof. Don’t forget conventions on omitting parentheses. The notation “NA” means “not assigned”.

NA-1. Explain the Russell and Whitehead “dot notation” used in Principia Mathematica. (See page 10 of volume 1 of Principia Mathematica which is available online from the University of Michigan, or see the entry in the online Stanford Encyclopedia of Logic on the notation used in the Principia Mathematica.) Specifically: describe how to convert from parentheses to dot notation, and vice-versa.

1. Prove that \{\land, \lor, \rightarrow\} is not a functionally complete set of connectives.

2. Prove that \{\neg, \leftrightarrow\} is not a functionally complete set of connectives. (For the latter, see also problem 7 below.)

3. Let the Sheffer stroke (NAND) be defined so that \( p \uparrow q \) means the same as \( \neg (p \land q) \). Prove that \{\uparrow\} is functionally complete.

4. Prove that \{\rightarrow, \bot\} is functionally complete.

5. Let \( \downarrow \) be the NOR connective; \( (p \downarrow q) \equiv \neg (p \lor q) \). Prove that there is no other binary Boolean connectives besides \( \downarrow \) and \( \uparrow \) that is functionally complete. [Hint: there are 16 binary Boolean connectives, including constant functions and functions that depend on only one input.]

6. Use a truth table to prove that \( (p \leftrightarrow q) \leftrightarrow r \) is tautologically equivalent to \( p \leftrightarrow (q \leftrightarrow r) \).

7. Describe succinctly the conditions under which the formula

\[ p_1 \leftrightarrow p_2 \leftrightarrow p_3 \leftrightarrow \cdots \leftrightarrow p_n \]
is true. That is, give a natural characterization of when a truth assignment assigns this formula the value True.

8. Let \( p_{i,j} \) be distinct variables, for \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \). Consider the CNF formula

\[
\varphi := \bigwedge_{i=1}^{m} \bigvee_{j=1}^{n} p_{i,j}.
\]

Describe the formula \( \varphi^{DNF} \) that is (a) equivalent to \( \varphi \) and (b) is in disjunctive normal form. (Say, as obtained from \( \varphi \) by using distributivity of \( \land \) and \( \lor \).) How large is the formula \( \varphi^{DNF} \)? (Suggestion: measure the size of \( \varphi^{DNF} \) by the number of occurrences of variables in the formula.)

Can you prove your bound on the size of the DNF formula is optimal?

9. Give an example of a set \( \Gamma \) of formulas which is not tautologically equivalent to any finite set \( \Delta \) of formulas.

10. Suppose that \( \Gamma \) is tautologically equivalent to some finite set \( \Delta \). Prove that there is a finite subset \( \Gamma_0 \) of \( \Gamma \) such that \( \Gamma \) and \( \Gamma_0 \) are tautologically equivalent.

11. Suppose that \( \Gamma \) and \( \Delta \) are sets of formulas that express the negation of each other. Namely, suppose that any given truth assignment \( \tau \) satisfies exactly one of \( \Gamma \) or \( \Delta \). Prove that \( \Gamma \) (and similarly \( \Delta \)) is tautologically equivalent to some finite set formulas.