Math 260AB - Introduction to Mathematical Logic

Winter and Spring 2012

Instructor: Sam Buss

Homework II. Due: Wednesday, February 8.

General instructions for all homeworks: Proofs are not required unless the problem asks for a proof. The notation "**NA**" means "not assigned".

1. Give an explicit Frege proof of $\varphi \to \psi \vdash \neg\psi \to \neg\varphi$.

2. Without using the completeness theorem for this or the next problem, prove the following have Frege proofs:

- $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
- $\neg(A \vee B) \to \neg A \wedge \neg B$
- $(A \to B \to C) \to (A \wedge B) \to C$.

3. Prove (for the Frege proof system) that $\vdash (A \vee \neg A)$.

4. The next two problems are for the sequent calculus. Prove (without using the Completeness Theorem) that the sequent $A \longrightarrow A$ has a proof, for arbitary formula A.

5. Give explicit sequent calculus proofs for the following sequents.

- $\longrightarrow (p \to q) \to (q \to r) \to (p \to r)$.
- $(p \wedge q) \vee r \longrightarrow (p \vee r) \wedge (q \vee r)$.
- $p \wedge q \longrightarrow \neg(\neg p \vee \neg q)$.

6. (Refutation of the 3-2 pigeonhole principle.) Let Γ be the following clauses:

$$\begin{aligned}\Gamma \;=\; & \{\{p_{11}, p_{12}\}, \{p_{21}, p_{22}\}, \{p_{31}, p_{32}\}, \{\overline{p}_{11}, \overline{p}_{21}\}, \{\overline{p}_{11}, \overline{p}_{31}\}, \\ & \{\overline{p}_{21}, \overline{p}_{31}\}, \{\overline{p}_{12}, \overline{p}_{22}\}, \{\overline{p}_{12}, \overline{p}_{32}\}, \{\overline{p}_{22}, \overline{p}_{32}\}\}.\end{aligned}$$

Give a resolution refutation of Γ.

7. Suppose $C \subset D$ are clauses. Suppose Γ, D has a resolution refutation of n lines in length. Prove that Γ, C has a resolution refutation in $\leq n$ lines in length.

8. (Strong completeness of resolution) Suppose that $\Gamma \vDash D$. Prove that there is a clause $C \subseteq D$ so that there is a resolution *derivation* of C from the hypotheses Γ.

9. Using the predicate symbols $L(x, y)$ for "x loves y", and $M(x, y)$ for "x is the mother of y", translate the following sentences into first-order logic. Use as simple a formula as possible, and use equality only if necessary.

a. Everyone has a mother.

b. Everyone has exactly one mother.

c. Every mother loves all her children.

d. No one loves everyone.

e. If someone loves everyone, then they are loved by everyone.

f. If someone loves everyone else, then they are loved by everyone else.

g. Abe loves everyone who does not love their mother.

10. Abraham Lincoln is sometimes claimed to have stated, "You can fool some of the people all of the time, and you can fool all of the people some of the time, but you cannot fool all of the people all of the time." Use variables named x to range over people, and variables named t to range over moments in time, and use $L(x, t)$ to mean "you can fool person x at time t". Express the three components of Lincoln's sentence in first order logic. If there is more than way natural way to express some of them in first order logic, explain what the various ways are and how they differ.

11. Consider the first order language with only equality ($=$) and no non-logical symbols. Give a first order statement expressing that there are exactly two distinct elements. More generally, for fixed $k \in \mathbb{N}$, explain how to write a first order sentence E_k expressing the condition that there are exactly k elements in the universe.

How many symbols are in your sentence E_k? Is it $O(k^2)$? Can you express it using only $O(k)$ symbols?

12. Continuing problem 11: Give a set Γ of first order sentences such that Γ is true exactly when there exist infinitely many objects.