Math 260AB - Introduction to Mathematical Logic

Winter and Spring 2012

Instructor: Sam Buss

Homework IV. Due: Thursday, March 22.

General instructions for all homeworks: Proofs are not required unless the problem asks for a proof. The notation "**NA**" means "not assigned".

The first two problems concern propositional logic. The first problem concerns propositional resolution.

Definition A *Horn clause* is a clause in which at most one variable appears unnegated. For instance $\{p, \overline{q}, \overline{r}\}$ and $\{\overline{q}, \overline{r}\}$ are both Horn clauses. Note that the meaning of the first clause is the same as $q \wedge r \to p$.

1. Suppose that Γ is a set of Horn clauses. Prove that either Γ is satisfiable, or that Γ has a unit resolution refutation.

2 . Let G be a fixed infinite graph. G is said to be 4-colorable if it is possible to assign each vertex in G one of four colors so that no two adjacent vertices are assigned the same color. Give a set Γ_G of propositional formulas such that Γ_G is satisfiable iff G is 4-colorable. The suggestion is to use propositional variables $p_{n,c}$ which mean that vertex n is assigned color c. Use this to prove that G is 4-colorable iff every finite subgraph of G is four colorable.

3.* Let L be a (infinite) language and $\mathcal{M}$ a finite model. Let T be the set of sentences which are true in $\mathcal{M}$. Prove or disprove: T is categorical. [Hint: You may wish to use compactness, but other proofs are possible too.] (This problem was erroneously first stated with "quantifier-free formulas" in place of "sentences". Partial credit for solving it as originally stated.)

4. Consider the valid formula $(\exists x)(\forall y)(P(y) \to P(x))$. By Herbrandization $(\exists x)(P(g(x)) \to P(x)$ is also valid. Find a finite set of terms $t_1, t_2, \ldots$ so that that $\bigvee_i (P(g(t_i) \to P(t_i)$ is tautologically valid.

5. Herbrand's theorem states that any valid $\forall\exists$ formula can be shown valid using some finite set of terms. What about $\exists\forall$ formulas? Is it the case that if $\vDash \exists x \forall y \varphi(x, y)$ is a valid sentence, then there exists a finite set of terms t_i such that $\bigvee_i \forall y \varphi(t_i, y)$ is valid? Either prove this, or give a counterexample.

6. (Quantifier elimination for Dense Linear Order without Endpoints.) Let T be the theory of dense linear ordering without endpoints as follows. Recall that T is complete, and is the same as the theory of $(\mathbb{Q}, <)$ and the theory of $(\mathbb{R}, <)$. [General hint: This means that when asked to show something is T-provable it is enough to have it be valid in either of these two structures.]

a. Define an *arrangement* of $x_1, \ldots, x_k$ to be a complete description of how the x_i's are ordered and whether any two x_i's are equal or unequal. For example, $x_1{<}x_2 \wedge x_2{=}x_3 \wedge x_3{<}x_4$ is an arrangement of x_1, x_2, x_3, x_4. Also allow the *inconsistent arrangement* $x_1 < x_1$ as an arrangement. Note that every arrangement is equivalent to a conjunction of atomic formulas. For each k, there are finitely many arrangements of $x_1, \ldots, x_k$.

b. Prove that every quantifier-free formula is T-provably equivalent to a disjunction of arrangements. [Hint: Use disjunctive normal form and induction on the complexity of formulas.]

c. Suppose $\varphi(x_1, \ldots, x_k)$ is an arrangement. Prove that $\exists x_k \varphi$ is T-provably equivalent to an arrangement of $x_1, \ldots, x_{k-1}$. [Hint: Describe how to obtain the arrangement of $x_1, \ldots, x_{k-1}$ from the arrangement of $x_1, \ldots, x_k$.

d. Suppose $\varphi(x_1, \ldots, x_k)$ is a disjunction of arrangements. Prove that $\exists x_k \varphi$ is T-provably equivalent to a disjunction of arrangements of $x_1, \ldots, x_{k-1}$. [Hint: This follows easily from c.]

e. Prove that every first-order formula φ is equivalent to a disjunction of arrangements, and hence to a quantifier-free formula. [Hint: induction on the complexity of formulas, and parts b. and d.]