Math 262A - Circuit Complexity

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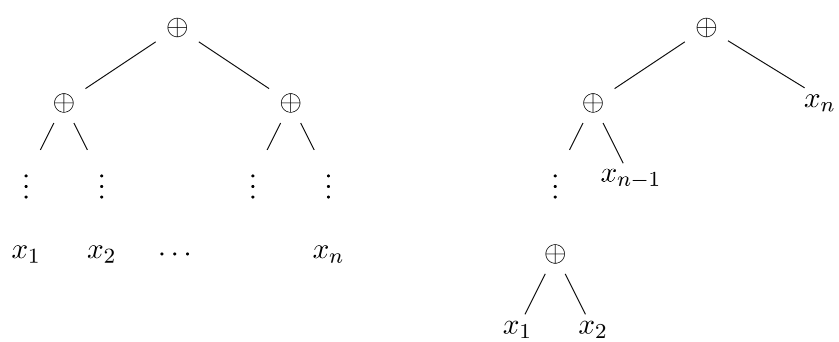
Lower Bounds on Formula Size

Recall the parity of n variables is a function such that $\text{Parity}(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i \bmod 2$. In Class 1, we have seen that the formula size (with $\oplus$ gates) of Parity is $n-1$, i.e., $\text{Parity}(x_1, \ldots, x_n) = x_1 \oplus x_2 \oplus \ldots, \oplus x_n$. As a circuit, it can ben constructed as many different binary trees, for example the following circuits are two possible trees,



construction (1) construction (2)

In this first construction, the depth of this circuit is $\log n$, however in the second construction, the depth is $n-1$. If we are going to find the boolean formula with only $\neg, \wedge, \vee$ gates, we can use the formula

$$x_1 \oplus x_2 = (x_1 \wedge \neg x_2) \vee (\neg x_1 \wedge x_2). \tag{1}$$

Definition 1. *Let ϕ be a formule with basis $\{\neg, \wedge, \vee\}$, we define $\star$-size(ϕ) to be the number of $\wedge, \vee$ gates that appear in ϕ.*

We define leafsize(ϕ) to be the number occurences of inputs in ϕ, and also define $\star$-depth(ϕ) to be the maximum number of $\wedge$ or $\vee$ gates in any path in ϕ from input to the output.

Property 2. $\text{leafsize}(\phi) = \star\text{-size}(\phi) + 1$.

Consider the parity function, $\text{Parity}(x_1, \ldots, x_n)$, and let $n = 2^i$. If we using the first construction (i.e., balance one) replacing $\oplus$ by (1), then we get a formula of $\star$-depth $2\log n = 2i$and with leaf size $2^{2i} = (2^i)^2 = n^2$.

Definition 3. *Let ϕ be a formula such that $\neg$ applies only to the inputs, let d be the maximum number of alternations of $\wedge' s + \vee' s$ on any path of ϕ, we define the $\wedge/\vee$ alternation depth of ϕ to be $d+1$.*

Notice that we can either use $x_1 \oplus x_2 = (x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2)$ or $x_1 \oplus x_2 = (x_1 \wedge \neg x_2) \vee (\neg x_1 \wedge x_2)$, so we can construct a formula for $\text{Parity}(x_1, \ldots, x_n)$ with $\wedge/\vee$ alternation depth $i+1$. Here we need to use De Morgan low to push the negations down, however it will not increases the $\wedge/\vee$ alternation depth.

Definition 4. *Let $\sigma, \tau \in \{0,1\}^n$ such that $\sigma \neq \tau$, we say that σ and τ are* neighbors *if the Hamming distance of them is* 1.

For two sets $A, B \subseteq \{0,1\}^n$, we set

$$N(A,B) := |\{(\sigma,\tau) \in A \times B : \sigma, \tau \text{ are neighbor}\}|.$$

Theorem 5 (Krapchenko Theorem)**.** *Let $f : \{0,1\}^n \to \{0,1\}$ be a function, and let $A \subseteq f^{-1}(0), B \subseteq f^{-1}(1)$, then*

$$L^*_{\neg,\wedge,\vee}(f) \geq \frac{N(A,B)^2}{|A| \cdot |B|} - 1.$$

Proof. Let ϕ be a formula computing f, we will show that

$$\text{leafsize}(\phi) \geq \frac{N(A,B)^2}{|A| \cdot |B|}.$$

We will prove it by induction on ϕ.

- **Base case:** $\phi = x_i$ for some x_i, i.e., no gate. In this case, we can show that $N(A,B) \leq |A|$, since for each $\sigma \in A$ there is at most one neighbor in B, and similar we can show that $N(A,B) \leq |B|$, the claim them follows.

- **Induction case 1:** $\phi = \neg\psi$ for some formula ψ. In this case, $\phi^{-1}(1) = \psi^{-1}(0)$ and $\phi^{-1}(0) = \psi^{-1}(1)$. Then for every $A \subseteq \phi^{-1}(0)$ and $B \subseteq \phi^{-1}(1)$, it has that $A \subseteq \psi^{-1}(1)$ and $B \subseteq \psi^{-1}(0)$, the claim then follows from induction.

- **Induction case 2:** $\phi = \psi \wedge \chi$ for some formule ψ, χ. In this case, it has that $A \subseteq \phi^{-1}(0) = \psi^{-1}(0) \cup \chi^{-1}(0)$ and $B \subseteq \phi^{-1}(1) = \psi^{-1}(1) \cap \chi^{-1}(1)$. Let $A_\psi \subseteq \psi^{-1}(0)$ and $A_\chi \subseteq \chi^{-1}(0)$ be two disjoint sets such that $A_\psi \cup A_\chi = A$, denote $n_\psi = N(A_\psi, B)$, $n_\chi = N(A_\chi, B)$, and $a_\psi = |A_\psi|$,

$a_\chi = |A_\chi|$, then

$$\begin{aligned}
\text{leafsize}(\phi) &= \text{leafsize}(\psi) + \text{leafsize}(\chi) \\
&\geq \frac{n_\psi^2}{a_\psi \cdot |B|} + \frac{n_\chi^2}{a_\chi \cdot |B|} \\
&= \frac{(n_\psi^2 a_\chi + n_\chi^2 a_\psi)(a_\psi + a_\chi)}{|B| \cdot a_\psi \cdot a_\chi (a_\psi + a_\chi)} \\
&= \frac{n_\psi^2 a_\psi a_\chi + n_\chi^2 a_\psi a_\chi + n_\psi^2 a_\chi^2 + n_\chi^2 a_\psi^2}{|B| \cdot a_\psi \cdot a_\chi (a_\psi + a_\chi)} \\
&\geq \frac{n_\psi^2 a_\psi a_\chi + n_\chi^2 a_\psi a_\chi + 2 n_\psi n_\chi a_\psi a_\chi}{|B| \cdot a_\psi \cdot a_\chi (a_\psi + a_\chi)} \\
&= \frac{(n_\psi + n_\chi)^2 a_\psi a_\chi}{|B| \cdot a_\psi \cdot a_\chi (a_\psi + a_\chi)} \\
&= \frac{N(A,B)^2}{|A| \cdot |B|},
\end{aligned}$$

where the last step we use the fact that $N(A,B) = N(A_\psi, B) + N(A_\chi, B)$.

- **Induction case 3:** $\phi = \psi \vee \chi$ for some formule ψ, χ. the proof here is similar to the case 2.

Based on the induction, the theorem then follows. □

This theorem has lots of applications, and we show two of them.

Claim 6. *Let $f(x_1, \ldots, x_n)$ be the parity function, then*

$$L^*_{\neg,\wedge,\vee}(f) \geq n^2.$$

Proof. Let $A = \{\sigma \in \{0,1\}^n : f(\sigma) = 0\}$ and $B = \{\tau \in \{0,1\}^n : f(\tau) = 1\}$. Then $|A| = |B| = 2^{n-1}$ and $N(A,B) = n|A|$, thus we have that

$$L^*_{\neg,\wedge,\vee}(f) \geq \frac{N(A,B)^2}{|A||B|} = n^2.$$

□

Claim 7. *Let $Th_{k,n}(x_1, \ldots, x_n)$ be the threshold function, i.e.*

$$Th_{k,n}(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } n \sum x_i \geq k \\ 0 & \text{otherwise} \end{cases}$$

then $L^*_{\neg,\wedge,\vee}(Th_{k,n}) \geq k(n-k+1) - 1$.

Proof. Let $A = \{\sigma \in \{0,1\}^n : \sigma \text{ has } k-1 \text{ many 1's}\}$ and $B = \{\tau \in \{0,1\}^n : \tau \text{ has } k \text{ many 1's}\}$, then $N(A,B) = k|B| = (n-k+1)|A|$, hence the claim follows. $\square$

Claim 8. *Let $Maj(x_1, \ldots, x_n)$ be the majority function, i.e.*

$$Maj(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } n \sum x_i \geq \frac{n}{2} \\ 0 & \text{otherwise} \end{cases}$$

then $L^*_{\neg,\wedge,\vee}(Maj) \geq n^2/4$.

Proof. Take $k = \lceil n \rceil /2$ in Claim 6, i.e.,

$$Maj(x_1, \ldots, x_n) = Th_{\lceil n \rceil, n/2}(x_1, \ldots, x_n)$$

thus $L^*_{\neg,\wedge,\vee}(Maj) = Th_{\lceil n \rceil, n/2}(x_1, \ldots, x_n) \geq n^2/4$. $\square$

Theorem 9 (Spira)**.** *Let C be a B_2 formula of* leafsize*m, then there is an equivalent formula C' over $\{\neg, \wedge, \vee\}$ such that*

- $\star$-depth$(C') \leq 2 \cdot \log_{3/2} m \approx 3.419 \log_2 m$;
- leafsize$(C') \leq m^\alpha$, *if* $\frac{1+2^\alpha}{3^\alpha} \leq 1/2$, *i.e.,* $\alpha = 2.196$.

Proof. The proof mainly based on rotating, it needs the following lemma.

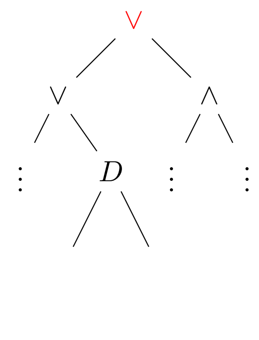
Lemma 10 ($1/3 \sim 2/3$ trick)**.** *Let T be a binary tree with m leaves, where $m \geq 1$. Then T has a subtree S such that m_S, number of leaves in S satisfy $\frac{1}{3}m \leq m_S \leq \frac{2}{3}m$.*

Proof. for $1/3 \sim 2/3$ trick. Take the minimum subtree S of T such that $\frac{1}{3}m \leq m_S$, then we claim that $m_S \leq \frac{2}{3}m$, otherwise either left subtree or right subtree of S will be a smaller subtree of T with leaf size greater or equal to $\frac{1}{3}m$. $\square$

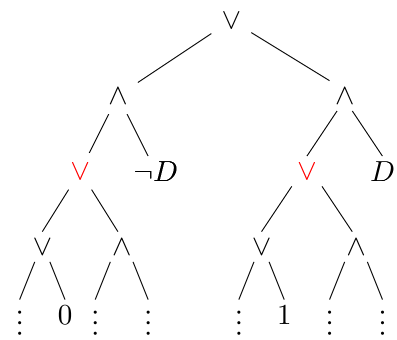
Then we finish this proof by induction on the leaf size of C.

- **Base case:** leafsize$(C) = 1$. In this case, we just set $C' = C$
- **Induction step:** $m =$ leafsize$(C) > 1$. By $1/3 \sim 2/3$ trick, there is a subformula D of C such that $\frac{1}{3}m \leq$ leafsize$(D) \leq \frac{2}{3}m$. Let's write the formula $C(x_1, \ldots, x_n, D(x_1, \ldots, x_n))$, then we can defind

$$C'(x_1, \ldots, x_n) := (C(x_1, \ldots, x_n, 0) \wedge \neg D) \vee (C(x_1, \ldots, x_n, 1) \wedge D),$$



original formula C



new formula C'

Then by induction, it has that

- $\star$- depth$(C') \leq 2 + 2\log_{3/2}\frac{2m}{3} = 2\log_{3/2} m$;
- leafsize$(C') \leq 2m_D^\alpha + 2(m - m_D)^\alpha \leq m^\alpha$.

The theorem them follows. $\square$