Monotone Formula for majority (Math 262A), Session 13

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Let's review the definition of Majority and Threshold first.

Definition 1. *Let $x_0, \ldots, x_{n-1}$ be input bits, we defind the threshold function as*

$$Th_k^n(x_0, \ldots, x_{n-1}) = \begin{cases} 1 & \textit{if } \sum x_i \geq k \\ 0 & \textit{otherwise} \end{cases}$$

and we also define the majority funcion as

$$Maj^n(x_0, \ldots, x_{n-1}) = Th^n_{\lceil n/2 \rceil}(x_0, \ldots, x_{n-1}).$$

Also we showed in the previous lecture that both Th_k^n and Maj^n are in NC^1, and also all of these functions are monotone boolean functions, thus it is natural to ask that

Could we prove that $Maj^n, Th_k^n \in$ monotone NC^1?

As the first stage of this, we can try the divide and conquer method, i.e.,

Definition 2. *We can represent the threshold function as*

$$Th_k^n(x_0, \ldots, x_{n-1}) = \bigvee_{l \leq \min\{k, n/2\}} \left(Th_l^{n/2}(x_0, \ldots, x_{n/2-1}) \wedge Th_{k-l}^{n/2}(x_{n/2}, \ldots, x_n)\right),$$

here we assumed that n is the power of 2.

Then what is the depth of these formula? In fact, we have that

Lemma 3. *With unbounded fan in $\wedge$'s and $\vee$'s, the depth of the formula above is $O(\log n)$.*

The lemma is easy to prove, however in the case of fan-in 2 $\wedge$'s and $\vee$'s, the depth is $O((\log n)^2)$ and the size is $2^{(\log n)^2} = n^{O(\log n)}$. Fortunately, we can also show that $Maj^n, Th_k^n \in$ monotone NC^1 by the following probabilistic method.

*Lectured on October 4, 2013, and scribed by Jiapeng Zhang

Theorem 4 (Valiant 1983). *Majority has monotone fan-in 2 formula of depth $O(\log n)$, hence size $n^{O(1)}$.*

Proof. Let $Maj(x_0, \ldots, x_{n-1})$ be the majority function, and without loss of generality, we assume that n is even. We consider the circuits that will contains subcircuits that compute $Z' = (Z_1 \vee Z_2) \wedge (Z_3 \vee Z_4)$, i.e.,

Definition 5. *A level 0 formula φ is seclected at random as*

$$\varphi = \begin{cases} x_i, & \text{with probability } \frac{2\alpha}{n-1} \\ 0 & \text{with probability } 1 - \frac{2\alpha n}{n-1} \end{cases}$$

where $\alpha = \frac{3-\sqrt{5}}{2} \approx 0.38$.

A level $i+1$ formula is selected at random by choosing level i formulas $\varphi_1, \ldots, \varphi_4$ at random, and setting $\varphi = (\varphi_1 \vee \varphi_2) \wedge (\varphi_3 \vee \varphi_4)$.

Lemma 6 (Valiant 83). *Let α as above, let $1 < \gamma < 4\alpha$, then we can choose $t = (1 + \frac{1}{(\log \gamma)^2}) \log n + O(1) = O(\log n)$ such that, for φ a randomly chosen level t formula, $\Pr[\varphi \equiv Maj(x_0, \ldots, x_{n-1})] \geq 1/2$.*

Lemma 7 (continue). *In fact, for values $a_0, \ldots, a_{n-1}$ as the inputs, then*

- *if $\sum_i a_i \geq n/2$, $\Pr_\varphi[\varphi(a_0, \ldots, a_{n-1}) = 1] \geq 1 - \frac{1}{2^{n+1}}$;*
- *if $\sum_i a_i < n/2$, $\Pr_\varphi[\varphi(a_0, \ldots, a_{n-1}) = 0] \geq 1 - \frac{1}{2^{n+1}}$;*

Lemma 6 follows from lemma 7 by a standard averaging argument, hence it is sufficient to prove lemma 7.

Proof. (*of lemma* 7). Fix $a_0, \ldots, a_{n-1} \in \{0, 1\}$, let $k = \sum_i a_i$, define

$$P_l := P_{l,k} = \Pr[\varphi(a_0, \ldots, a_{n-1}) = 1],$$

where φ is randomly chosen level l formula. Then we are going to show that

$P_t \geq 1 - \frac{1}{2^{n+1}}$, if $k \geq n/2$;

$P_t \leq \frac{1}{2^{n+1}}$, if $k < n/2$.

By the definition, we have that $P_0 = \frac{2\alpha k}{n-1}$ and

$$P_{i+1} = (1 - (1 - P_i)^2)^2 = 4P_i^2 - 4P_i^3 + P_i^4.$$

Let $f(x) = 4x^2 - 4x^3 + x^4$, then we have

$f(0) = 0$, $f(\alpha) = \alpha$, and $f(1) = 1$;

$f'(0) = 0$, $f'(\alpha) = 4\alpha$, and $f'(1) = 0$;

$f''(0) = 8$, and $f''(1) = -4$;

Also, by the definitions

If $k < n/2$, then

$$P_0 \leq \frac{2\alpha(\frac{n}{2}-1)}{n-1} = \frac{\alpha(n-2)}{n-1} < \alpha - \frac{\alpha}{n};$$

Similarly, if $k \geq n/2$, then

$$P_0 \geq \alpha + \frac{\alpha}{n},$$

here we used the fact that n is even.

By continuity of the first derivative, for any $1 < \gamma < 4\alpha$, $\exists \epsilon_0$ such that if $|P_i - \alpha| < \epsilon_0$, then

$$|f(P_i) - \alpha| \geq |P_i - \alpha| \cdot \gamma.$$

Choose $l_1 = \log_\gamma(n \cdot \epsilon_0/2) = \log_\gamma(n) = O(\log n)$, then $\frac{2}{n}\gamma^{l_1} \geq \epsilon_0$, so

if $k < \frac{n}{2}$, then $P_{l_1} < n - \epsilon_0$

if $k \geq \frac{n}{2}$, then $P_{l_1} > n + \epsilon_0$,

where ϵ_0 is a constant that does not depend on n.

So we can take $l_2 = l_1 + c$, where c is a constant such that

if $k < \frac{n}{2}$, then $P_{l_2} < \frac{1}{16}$

if $k \geq \frac{n}{2}$, then $P_{l_2} > 1 - \frac{1}{8} = \frac{7}{8}$,

From more calculus facts, since $f(x) \leq 8x^2$ and $(1 - f(1-x)) \leq 4x^2$ for the range $x \in [0, 1]$ we have that at some point the two cases begin to diverge at a quadratic rate. Thus:

if $k < \frac{n}{2}$, let $Q_i = 8P_i$. Since $P_{i+1} \leq 8P_i^2$, then $Q_{i+1} \leq Q_i^2$. Also $Q_{l_2} \leq 1/2$, we have that $Q_{l_2+\log n+3} \leq 2^{-(n+4)}$, thus

$$P_{l_2+\log n+3} \leq 2^{-(n+1)}$$

if $k < \frac{n}{2}$, let $Q_i = 4(1 - P_i)$. Since $P_{i+1} \leq 8P_i^2$, so $Q_{i+1} \leq Q_i^2$ and $Q_{l_2} \leq 1/2$, thus $Q_{l_2+\log n+2} \leq 2^{-(n+3)}$, thus

$$P_{l_2+\log n+2} \geq 1 - 2^{-(n+1)}$$

Let $l = l_2 + \log n + 3$, then lemma 7 then follows. □

The theorem also follows from the lemma 6. □