Math260/267 - Set Theory

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Part I
Course Outline

1 Introduction

Properties versus sets. The Collection principle. Russell's paradox. Separation versus collection. Sets versus classes.

2 The First-order Language of Set Theory

The language of set theory contains the following basic symbols.

Boolean connectives: $\neg$, $\rightarrow$, $\wedge$, $\vee$, $\leftrightarrow$.

Quantifiers: $\forall$ and $\exists$

Variables: $x_1, x_2, x_3, \ldots,$, also denoted $x, y, z, \ldots$. Variables range over the universe of all sets.

Equality: $=$

Membership: $\in$

Parentheses: (,)

All (or nearly all) set-theoretic assertions we will ever make are expressible in the above first-order language. Of course, we will introduce many abbreviations to avoid having to write very long statements in the base first-order language.

3 Axioms of Set Theory

The axioms for Zermelo-Fraenkel set theory, ZF, are:

Logical Axioms and Equality Axioms: These are axioms that state the basic properties of the Boolean connectives, the quantifiers, and the equality symbol. This includes the assertion that some set exists.

Extensionality: A set is determined by its members:

$$\forall x \forall y[\forall w(w \in x \leftrightarrow w \in y) \rightarrow x = y].$$

Pair Set Axiom: For all x, y, $\{x, y\}$ exists:

$$\forall x \forall y \exists z[\forall w(w \in z \leftrightarrow w = x \vee w = y)].$$

Union: For all x, $\bigcup x$ exists:

$$\forall x \exists z[\forall w(w \in z \leftrightarrow \exists y(y \in x \wedge w \in y))].$$

Separation Axioms: For every first-order formula $\varphi(x, \vec{y})$ and every u, $\vec{y}$, the set $\{x : x \in u \wedge \varphi(x, \vec{y})\}$ exists:

$$\forall u \forall y_1 \cdots \forall y_k \exists z[\forall x(x \in z \leftrightarrow x \in u \wedge \varphi(x, \vec{y}))].$$

Note there are infinitely many separation axioms.

Axiom of Infinity: There exists an infinite set:

$$\exists x(0 \in x \wedge \forall y(y \in x \rightarrow S(y) \in x)).$$

Power Set Axiom: For all x, the power set $\wp(x)$ of x exists:

$$\forall x \exists z \forall w(w \in z \leftrightarrow w \subset x).$$

Replacement Axioms: The image a of definable function on a set w exists. Let $\varphi(x, z, \vec{y})$ be a first-order formula.

$$\forall \vec{y} \forall w[((\forall x \in w)\exists! z \varphi(x, z, \vec{y})) \rightarrow \exists s((\forall x \in w)(\exists z \in s)\varphi(x, z, \vec{y}))].$$

There are infinitely many instances of the replacement axiom.

Regularity (Axiom of Foundation): Every non-empty set contains an $\in$-minimal element:

$$\forall x(x \neq \emptyset \rightarrow \exists y(y \in x \wedge y \cap x = \emptyset)).$$

3.1 Comments on the above axioms

Examples of how abbreviations such as $\supset$, $\subset$, $\emptyset$, $\{x, y\}$, $\bigcup x$, $\bigcap x$, etc., may be used to help with writing and reading first-order formulas.

Ordered Pairs, ordered triples, ordered n-tuples.

Natural Numbers. $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, etc. $\omega = \{0, 1, 2, \ldots\}$.

Successor: $S(x) = x \cup \{x\}$.

ω exists and is uniquely defined.

Definition A set x is *inductive* iff $0 \in x$ and $(\forall y \in x)(S(y) \in x)$.

Definition We use the abbreviation $\exists! x$ for "there exists a unique x". Thus, $\exists! x \varphi(x)$ is an abbreviation for

$$\exists x \varphi(x) \wedge \forall x \forall y (\varphi(x) \wedge \varphi(y) \rightarrow x = y)$$

The Power Set Axiom can be used to prove that the crossproduct of two sets is a set. Later, we'll see that the Replacement Axiom can be used for this instead.

Definition From cross-products, we can define *binary relations*, *functions*, the *range*, *domain* and *field* of a function.

Definition A *countably infinite sequence* is a function with domain ω. Such a function f is (i.e., "represents") the sequence $(x_0, x_1, x_2, \ldots)$ where $x_i = f(i)$.

A *finite sequence* of length n, or a n-tuple, is a function with domain n.

Theorem 1 *Let A and B be sets. The set* $\{f : f \text{ is a function from } A \text{ to } B\}$ *exists.*

The proof (necessarily) uses the Power Set Axiom.

Definition The notions of *partial order*, *(linear) order*, *strict order*, and *well-order* are defined as usual.

4 The Natural Numbers ω

Earlier, we characterized ω as the intersection of all inductive sets. This definition immediately tells that that the induction principle holds for the natural numbers:

Theorem 2 **(Integer Induction)** *Let $\varphi(x, y_1, \ldots, y_k)$ be a first-order formula. Let $y_1, \ldots, y_k$ be fixed sets. Suppose that $\varphi(0, \vec{y})$ is true and that $(\forall x \in \omega)(\varphi(x, \vec{y}) \to \varphi(S(x), \vec{y}))$. Then $\varphi(x, \vec{y})$ is true for all $x \in \omega$.*

Definition A set x is *transitive* if every member of x is a subset of x. This is equivalent to the statement that whenever $z \in y \in x$, then $z \in x$.

Theorem 3 *Let $x \in \omega$.*

(a) *x is transitive and every member of x is transitive.*

(b) *x is strictly well-ordered by $\in$.*

(c) $0 \in x$.

(d) *For all $y \in x$, either $S(y) \in x$ or $y = x$.*

(e) $x \subset \omega$.

(f) *If $x \neq 0$, then there is a y s.t. $x = S(y)$.*

Theorem 4 *ω is strictly well-ordered by ϵ.*

Theorem 5 *There is a formula $SumOf(x, y, z)$ which is true for exactly those integers x, y, z which satisfy $x + y = z$. This defines a unique z for for each $x, y \in \omega$, and satisfies the recursive definition of $+$ in terms of 0 and successor (S).*

Left as homework are similar theorems for multiplication and exponentiation. Once have definitions for addition, multiplication, and exponentiation, we can define many natural set of the non-negative integers. For example, there is a formula defining the primes, a formula expressing Fermat's Last Theorem. Furthermore, assertions such as the prime factorization can be stated in the first-order language of set theory and proved from the axioms of set theory (as can *any* other currently established fact about the integers, apart from some generally unnatural statements that depend on the consistency of set theory).

5 Ordinals

Ordinals provide a transfinite generalization of the integers. To be precise, it provides a generalization of the notion of "counting integers", e.g., a transfinite extension of the notions of "first", "second", "third", ...

5.1 Transitive Sets

In the previous section, we defined the notion of transitive set. This is a fundamental notion needed for the definition of ordinals.

Definition The *transitive closure*, $TC(x)$, of a set x is equal to

$$x \cup \left(\bigcup x\right) \cup \left(\bigcup\bigcup x\right) \cup \left(\bigcup\bigcup\bigcup x\right) \cup \cdots$$

Theorem 6 $TC(x)$ *is well-defined.*

Proof Define $\bigcup^i x$ by induction on $i \in \omega$ by $\bigcup^0 x = x$ and $\bigcup^{i+1} x = \bigcup\left(\bigcup^i x\right)$. By replacement, the set

$$z = \left\{\bigcup^i x : i \in \omega\right\}$$

exists. Then $TC(x) = \bigcup z$. □

Theorem 7 $TC(x)$ *is transitive.*

5.2 Basics of Ordinals

Definition An *ordinal* is set which is (a) transitive and (b) strictly well-ordered by $\in$.

We use Greek letters $\alpha, \beta, \gamma, \ldots$ for ordinals. We sometimes write $\alpha < \beta$ for $\alpha \in \beta$.

Examples of ordinals include the finite integers and the set ω.

Theorem 8 *If α is an ordinal, then $S(\alpha)$ is an ordinal.*

Theorem 9 *If α is an ordinal and $\beta \in \alpha$, then β is an ordinal.*

Lemma 10 *If α is an ordinal, $\alpha \notin \alpha$.*

Definition A subset X of α is an *initial segment* of α if and only if, for all $\beta \in \alpha$, either $\beta in X$ or $(\forall \gamma \in X(\gamma < \beta)$.

Theorem 11 *If α is an ordinal and A is an initial segment of α, then either $A \in \alpha$ or $A = \alpha$.*

Theorem 12 *If α and β are ordinals, then either $\alpha = \beta$ or $\alpha < \beta$ or $\beta < \alpha$.*

Definition **ON** is the class of all ordinals.

Definition A *class* C is a collection of sets which can be defined as follows: for some first-order formula $\varphi(x, y_1, \ldots, y_k)$ and some fixed sets $y_1, \ldots, y_k$,

$$C = \{x : \varphi(x, y_1, \ldots, y_k)\}.$$

A good example of a class is the class V of all sets, defined by $V = \{x : x = x\}$. V is called the *universe*. The class **ON** is defined by

$$\mathbf{ON} = \{x : \text{“}x \text{ is an ordinal”}\}.$$

The term "collection" in the previous definition refers to some intuitive notion of collection, or gathering together. Note that some classes are sets (e.g., the empty class), and that some classes are not sets (e.g., the class V of all sets).

Theorem 13 *The class* **ON** *is strictly well-ordered by the relation $\in$. In fact, any non-empty subbf class of* **ON** *has a $\in$-least element.*

Theorem 14 **ON** *is not a set. There is no set containing all the ordinals.*

Definition Let x be a set of ordinals. The supremum of x, $sup(x)$, is defined to equal $\cup x$.

Theorem 15 *If x is a set of ordinals, then $sup(x)$ is an ordinal.*

Theorem 16 *Let x be a set of ordinals. The $sup(x)$ is the unique least upper bound of x. I.e., for all $y \in x$, $sup(x) > y$ and $sup(x)$ is the least ordinal with this property.*

Examples: The finite integers are ordinals, ω is an ordinal, $\omega{+}1$, $\omega{+}2, \ldots$, are ordinals. $\omega \cdot 2 = \omega + \omega = sup\{\omega + n : n \in \omega\}$ is an ordinal. Also: $\omega \cdot n$ and $\omega^2 = \omega \cdot \omega$ are ordinals.

Definition An ordinal α is a *successor* ordinal if $\alpha = S(\beta)$ for some β. β is called the *predecessor* of x. If α is not a successor and $\alpha \neq 0$, then α is a *limit ordinal.*

5.3 Well-orderings and ordinals.

Theorem 17 *Let $(X, <)$ specify a set X well-ordered by $<$. There is a unique ordinal α which is order-isomorphic to $(X, <)$.*

Lemma 18 *Any order-isomorphism of f with domain X and range an initial segment of the ordinals satisfies*

$$f(x) \; = \; \sup\{f(y)+1 : y \in X, y \in X\}.$$

5.4 Transfinite Induction and Recursion

Transfinite Induction: Let $\varphi(\alpha, y_1, \ldots y_k)$ be a first-order formula and let $y_1, \ldots, y_k$ be fixed sets. Suppose that

$$(\forall \alpha \in \mathbf{ON})[((\forall \beta < \alpha)\varphi(\beta, \vec{y})) \rightarrow \varphi(\alpha)].$$

Then,

$$(\forall \alpha \in \mathbf{ON})\varphi(\alpha, \vec{y})$$

holds.

A special case of transfinite induction is induction up to a particular ordinal. This is obtained by replacing "**ON**" in the above by an arbitrary ordinal γ.

Frequently we use the following *special case of transfinite induction*: We prove

0. that $\varphi(0, \vec{y})$ holds,

1. that for all α, if $\varphi(\alpha, \vec{y})$, then $\varphi(\alpha + 1, \vec{y})$ holds, and

2. that for all limit ordinals α, if $(\forall \beta < \alpha)\varphi(\beta, \vec{y})$, then $\varphi(\alpha, \vec{y})$.

From this we may conclude that $\varphi(\alpha, \vec{y})$ holds for all ordinals α.

Transfinite Recursion Let $\varphi(x, z, y_1, \ldots, y_k)$ be a formula which defines a function $g(x) = z$; i.e., assume that $\forall x \exists ! z \varphi(x, z, \vec{y})$ is valid. Fix sets $y_1, \ldots, y_k$. Let f be defined by transfinite recursion by

$$f(\alpha) \; = \; g(f_{\restriction \alpha}),$$

where $f_{\restriction \alpha}$ means the set of ordered pairs $(\beta, f(\beta))$ for $\beta < \alpha$.

Theorem 19 *The above definition of* f *by transfinite recursion is a valid definition of a function. In particular, there is a formula* $\varphi(\alpha, z, \vec{y})$ *which defines* $f(\alpha) = y$. *The assumption that* φ *defines a function, implies that* $\forall\alpha \in \mathbf{ON}\exists! z\psi(x, z, \vec{y})$.

Frequently we use the following *special case of transfinite recursion*: We define $f(0, \vec{y})$, we define $f(\alpha+1, \vec{y})$ in terms of $f(\alpha\vec{y})$, and for limit ordinals α, we define $f(\alpha, \vec{y})$ in terms of the values of $f(\beta, \vec{y})$ for $\beta < \alpha$.

5.5 Ordinal Arithmetic

Definition The sum of two ordinals is defined using transfinite induction as:

1. $\alpha + 0 = \alpha$.
2. $\alpha + S(\beta) = S(\alpha + \beta)$.
3. $\alpha + \beta = \sup\{\alpha + \gamma : \gamma < \beta\}$ for β a limit ordinal.

Ordinal addition can also be characterized in terms of concatenation of well-orderings. Let $(X, <)$ and $(Y, <')$ be two well-ordered sets, and define their concatenation to be the well-ordering which consists of $(\{0\} \times X) \cup (\{1\} \times Y)$ with the lexicographic ordering. Then we have:

Theorem 20 $\alpha + \beta$ *is equal to the order type of the concatenation of* $(\alpha, \in)$ *and* $(\beta, \in)$.

Ordinal addition in not commutative. For example, $1 + \omega = \omega$ (as can easily be seen either from the induction definition of addition or from the characterization of ordinal addition in terms of concatenation). Thus $1 + \omega \neq \omega + 1$.

Definition The product of two ordinals is defined using transfinite induction as:

1. $\alpha \cdot 0 = 0$.
2. $\alpha \cdot S(\beta) = \alpha \cdot \beta + \alpha$.
3. $\alpha \cdot \beta = \sup\{\alpha \cdot \gamma : \gamma < \beta\}$ for β a limit ordinal.

We can characterize ordinal multiplication in terms of crossproducts. If X and Y are well-ordered as above, then we well-order $X \times Y$ anti-lexicographically so that (x, y) is less than (x', y') iff either $y <' y'$ or $y = y'$ and $x < x'$. Then we have:

Theorem 21 *$\alpha \cdot \beta$ is equal to the order type (otype) of $\alpha \times \beta$ under the anti-lexicographic ordering.*

Thus we think of $\alpha \cdot \beta$ as consisting of β many copies of α.

It is easy to see that $2 \cdot \omega = \omega \neq \omega \cdot 2$.

Definition The ordinal exponentiation is defined using transfinite induction as:

1. $\alpha^0 = 1$.
2. $\alpha^{S(\beta)} = \alpha^\beta \cdot \alpha$.
3. $\alpha^\beta = \sup\{\alpha^\gamma : \gamma < \beta\}$ for β a limit ordinal.

Examples include $\omega^0 = 1$, $\omega^1 = \omega$, ω^2, ω^3, ω^ω, ω^{ω^ω}, etc.

Definition We define (temporarily) ω_i to equal an exponential stack of i $\omega's$; i.e., $\omega_1 = \omega$ and $\omega_{i+1} = \omega^{\omega_i}$.

We define ϵ_0 to equal $sup\{\omega_i : i \in \omega\}$.

Theorem 22 $\omega^{\epsilon_0} = \epsilon_0$.

Any ordinal α which satisfies $\omega^\alpha = \alpha$ is called an *ϵ-number*. There is a homework problem with more properties of ϵ-numbers.

5.6 Cantor Normal Form

Theorem 23 *For every ordinal $\alpha > 0$, there exists a unique sequence of ordinals $\beta_1 \geq \beta_2 \geq \beta_3 \geq \cdots \beta_k$ (with $1 \leq k$) such that*

$$\alpha = \omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_k}.$$

In class we proved the uniqueness of the Cantor Norm Form representation of ordinals. The existence was left as a homework assignment. One fact needed for the uniqueness was:

Lemma 24 *If $\alpha < \beta$, then $\omega^\alpha + \omega^\beta = \omega^\beta$.*

The Cantor Normal Form Theorem provides us with a method of representing ordinals $< \epsilon_0$ with explicit definitions called *ordinal notations*. Some examples of these ordinal notations are:

$$0, \quad 1, \quad \omega, \quad \omega \cdot 2, \quad \omega^2, \quad \omega^3 + \omega + 3$$

and more complicated examples include

$$\omega^{\left(\omega^2+\omega\cdot 3+1\right)} + \omega, \quad \omega^{\omega^{\omega+\omega}+\omega^2+1}$$

These ordinal notations are just explicit representations for ordinals which are written using symbols "ω", "^" (exponentiation), "$\cdot$", "$+$", and integers from ω. In fact, we can get by without allowing "$\cdot$" and with allowing only the integer 0, by using the fact that $\omega^0 = 1$ and writing non-zero integers as a repeated sum of 1.

Formally, we can define the ordinal notations for ordinals less than ϵ_0 recursively by defining that 0 is an ordinal notation for the ordinal 0, and that if $\alpha_1, \ldots, \alpha_k$ are ordinal notations representing a non-increasing sequence of ordinals, then

$$\omega^{\alpha_1} + \cdots + \omega^{\alpha_k}$$

is an ordinal notation.

Theorem 25 *Every ordinal less than ϵ_0 has a unique ordinal notation.*

This theorem is basically an immediate consequence of the Cantor Normal Form Theorem, using the fact the $\omega^\alpha > \alpha$ for all $\alpha < \epsilon_0$ and using induction on ordinals.

The following theorem implicitly contains an effective, feasible algorithm, which given two (distinct) ordinal notations, determines which ordinal is less the other.

Theorem 26 *Let α and β have Cantor Normal forms*

$$\alpha \ = \ \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_k}$$

and

$$\beta \ = \ \omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_\ell},$$

then $\alpha < \beta$ if and only if either (a) $\alpha_1 < \beta_1$ or (b)$\alpha_1 = \beta_1$ and $\omega^{\alpha_2} + \cdots + \omega^{\alpha_k}$ is less than $\omega^{\beta_2} + \cdots + \omega^{\beta_\ell}$.

5.7 Goodstein Sequences

Integers can be written in *extended base k notation* by writing them in base k in the usual way, and then (recursively) writing the exponents of k in extended base k notation. For example, the integer 66048 can be expressed in extended base 2 notation as

$$66048 \;=\; 2^{2^{2^2}} + 2^{2^{2+1}+1}$$

and one can eliminate the use of 1 by using $1 = 2^0$. More formally we define:

Definition Let $n \in \omega$. The *extended base k notation* for n is inductively defined as follows. The expression 0 is the extended base k notation for $n = 0$. For $n > 0$, write n in base k notation as

$$n \;=\; k^{n_1} \cdot a_1 + \cdots + k^{n_\ell} \cdot a_\ell$$

where $n_1 > n_2 > \cdots > n_\ell$, and where $0 < a_i < k$ for all i. let $e_1, \ldots, e_\ell$ be the extended base k expressions for $n_1, \ldots, n_\ell$. Then the extended base k expression for n is

$$n \;=\; k^{e_1} \cdot a_1 + \cdots + k^{e_\ell} \cdot a_\ell.$$

We denote n's extended base k representation by $G_k(n)$.

Definition The expression $G_k(n)[p]$ is obtained from $G_k(n)$ by replacing each occurrence of k by p. We shall use this only with $p = k+1$ and with $p = \omega$. In the latter case, we obtain the ordinal notation of an ordinal less than ϵ_0.

Definition Let $n \in \omega$. Define $n_1 = n$ and

$$n_k = G_k(n_{k-1})[k+1] - 1$$

for all $k \geq 2$.

Theorem 27 (Goodstein) *For all $n \geq 2$, there exists an $i > 0$ such that $n_i = 0$.*

The idea of the proof involves proving the ordinals $G_k(n)[\omega]$ form a monotonically decreasing sequence.

Theorem 28 *The Goodstein Theorem cannot be proved in the system ZF without using the Axion of Infinity.*

The non-provability of the Goodstein Theorem without the Axiom of Infinite is quite remarkable since the Goodstein Theorem is a completely finitary proposition. The system of ZF minus the Axiom of Infinite is proof-theoretically equivalent to Peano Arithmetic.

6 The Axiom of Choice and Some Equivalents

The Axiom of Choice. The Well-Ordering Principle. Zorn's Lemma. The Hausdorff Maximal Principle.

Definition The *Axiom of Choice*, often referred to as AC, is the statement that every set of non-empty subsets has a choice function, i.e., if I is a set and if every $x \in I$ is non-empty, then there is a function f such that $domain(f) = I$ and such that $f(x) \in x$ for all $x \in I$.

Definition A variation on the axiom of choice, denoted AC', says that if $\{x_i\}_{i \in I}$ is a set of non-empty sets, then there is a choice function f with domain I s.t. for all $i \in I$, $f(i) \in x_i$.

Definition The *Well-Ordering Principle* is the first-order statement which states that every set can be well-ordered, i.e., for every set x there is a set of ordered pairs which is a well-ordering of x.

Theorem 29 *(AC)$\Longrightarrow$(Well-Ordering Principle)*

Theorem 30 *(Well-Ordering Principle)$\Longrightarrow$(AC).)*

Definition Let $(X, \leq)$ be a partially ordered set. A *chain* in X is a set of pairwise $\leq$-comparable members of X.

Let $Y \subset X$. u is an *upper bound* of Y if $u \in X$ and if $y \leq u$ for all $y \in Y$.

Definition *Zorn's Lemma* states that if $(X, \leq)$ is a partial order and if every chain in x has an upper bound, then X has a maximal element.

Theorem 31 *Zorn's Lemma is equivalent to the Axiom of Choice.*

Definition The *Hausdorff Maximal Principle* states that if $(X, \leq)$ is a partial order then every chain in X can be extended to a maximal chain in X.

Theorem 32 *The Hausdorff Maximal Principle is equivalent to the Axiom of Choice.*

7 Integers, Rationals and Reals

Definition of the integers, $\mathbb{Z}$, as pairs of non-negative integers under an equivalence relation.

Definition of the rationals, $\mathbb{Q}$, as pairs from $\mathbb{Z} \times \mathbb{N}^+$ under an equivalence relation

Definition of the reals as Dedekind cuts in $\mathbb{Q}$. The completeness theorem for the reals. The equivalent definition of the reals as equivalence classes of Cauchy sequences.

Theorem 33 *(*Cantor) *Every countable, unbounded, dense linear order is order-isomorphic to the rationals.*

Corollary 34 *Every complete, separable, dense, unbounded linearly ordered set is order-isomorphic to the reals.*

Definition A topological space satisfies the *countable chain condition (c.c.c)* iff does not have an uncountable collection of disjoint open sets.

Suslin's Problem *If* $(X, \leq)$ *is a complete, dense, unbounded, linearly ordered set that satisfies the countable chain condition, must* $(X, \leq)$ *be order-isomorphic to the reals?*

8 Cardinals

8.1 Introduction to Cardinals

We first give the definitions of $|x| \leq |y|$ and $|x| = |y|$ and $|x| < |y|$. Only afterward will we define the cardinals $|x|$, $|y|$ themselves.

Definition Let x and y be sets. We define $|x| \leq |y|$ to hold iff there is an injective (1-1) function $f : x \to y$. We define $|x| = |y|$ is hold iff there is a 1-1 correspondence (i.e., a 1-1 and onto function) $f : x \to y$.

$|x| < |y|$ iff $|x| \leq |y|$ and $|x| \neq |y|$.

Theorem 35 *If* $|x| \leq |y|$*, then there is a surjective (onto) map* $f : y \to x$*.*

Theorem 36 (AC) *If there is a surjective map* $f : y \to x$*, then* $|x| \leq |y|$*.*

Theorem 37 (Schröder-Bernstein Theorem) *If* $|x| \leq |y|$ *and* $|y| \leq |x|$*, then* $|x| = |y|$*.*

Definition *(AC)* Let x be a set. The cardinality, $|x|$, of x is the least ordinal α such that $|x| = |\alpha|$, i.e., such that there is a 1-1 correspondence between x and α.

Theorem 38 (AC) *For every set x, there is an ordinal α such that $|x| = |\alpha|$.*

Definition A *cardinal* is an ordinal α such that $|\alpha| = \alpha$.

We typically use Greek letters κ, λ, μ, etc. to denote for cardinals. (Recall that α, β, γ typically denote ordinals.)

Theorem 39 *Let α and β be ordinals and let $\kappa = |\alpha|$. If $\kappa \le \beta \le \alpha$, then $|\beta| = \kappa$.*

Definition A set x is *finite* iff $|x| \in \omega$. A set x is *countable* iff $|x| \le \omega$. A set x is *uncountable* iff it is not countable. A set x is *infinite* if it is not finite.

Definition A set x is Dedekind infinite iff it has a proper subset y such that $|x| = |y|$. If x is not Dedekind infinite, then it is *Dedekind finite.*

Theorem 40 *If A is a countably infinite set, then $A \times A$ is countably infinite.*

(AC) *The union of a countable set of countable sets is countable.*

Theorem 41 (Cantor) *If x is any set, then $|\wp(x)| \not\le |x|$.*

Corollary 42 *For all ordinals α, there exists a cardinal $\kappa > \alpha$.*

We proved this theorem in class using the Axiom of Choice, however, as a homework problem, you can prove it without using the Axiom of Choice.

Corollary 43 *There is no set of all cardinals. I.e., the cardinals form a proper class.*

Theorem 44 (AC) *Every infinite cardinal is a limit ordinal.*

8.2 Alephs

Definition An *aleph* is an infinite ordinal which is a cardinal.

We inductively define the β-th aleph by letting

$$\alpha_\beta \ = \ \text{the least cardinal} > \alpha_\alpha \ \text{for all} \ \alpha < \beta.$$

Theorem 45 *$\aleph_\alpha$ is defined for all $\alpha \in \mathbf{ON}$.*

Theorem 46 *The function $\beta \mapsto \aleph_\beta$ is continuous, i.e., for all limit ordinals β, $\aleph_\beta$ is equal to* $\sup\{\aleph_\alpha : \alpha < \beta\}$.

We also define ω_β to equal $\aleph_\beta$: the notation ω_β is used for its role as an ordinal, and the notation $\aleph_\beta$ is used for its role as a cardinal.

Definition Let $f : \mathbf{ON} \to \mathbf{ON}$ be a non-decreasing function. The function f is *continuous* iff for all limit ordinals β,

$$f(\beta) \ = \ \sup\{f(\alpha) : \alpha < \beta\}.$$

8.3 Cardinal Arithmetic

Definition A *limit cardinal* (respectively, a *successor cardinal*) is a cardinal of the form $\aleph_\beta$ for β a limit ordinal (resp., a successor ordinal).

Definition If κ is an ordinal, then κ^+ denotes the *cardinal successor* of κ, i.e., the least cardinal strictly greater than κ.

Definition The Continuum Hypothesis (CH) states that $2^{\aleph_0} = \aleph_1$.

The Generalized Continuum Hypothesis (GCH) states that $2^{\aleph_\beta} = \aleph_{\beta+1}$ for all ordinals β.

Definition Cardinal addition, multiplication and exponentiation (not to be confused with the ordinal operations!) are defined by

$$\begin{aligned}
\kappa + \lambda &= |\{(\{0\} \times \kappa) \cup (\{1\} \times \lambda)\}| \\
\kappa \cdot \lambda &= |\kappa \times \lambda| \\
\kappa^\lambda &= |{}^\lambda\kappa|,
\end{aligned}$$

where ${}^A B = \{f : A \to B\}$.

Theorem 47 *If κ is an infinite cardinal, then $\kappa + \kappa = \kappa$. Therefore, if at least one of κ or λ is infinite, then $\kappa + \lambda = \max(\kappa, \lambda)$.*

Theorem 48 *If κ is an infinite cardinal, then $\kappa \cdot \kappa = \kappa$. Therefore, if at least one of κ or λ is infinite, then $\kappa \cdot \lambda = \max(\kappa, \lambda)$.*

Theorem 49 $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$.

Proposition 50 *If $\mu \le \kappa$ and $\nu \le \lambda$, then $\mu^\nu \le \kappa^\lambda$.*

Theorem 51 *For $\kappa \ge \omega$, $2^\kappa = \kappa^\kappa$.*

8.4 Cofinality. Regular and Singular Cardinals

Definition Let $A \subset \beta$. We say A is *cofinal* in β iff $sup'(A) = \beta$.

An ordinal α is cofinal in β iff there is a function f such that $f : \alpha \to \beta$ and f is strictly increasing and the range of f is cofinal in β.

Definition The *cofinality*, $cf(\beta)$, of an ordinal β is equal to the least ordinal α which is cofinal in β.

Some examples include:

(a) Every ordinal is cofinal in itself. Therefore, $cf(\kappa) < \kappa$.

(b) For $n \in \omega$, $cf(n) = 1$. However, we are generally only interested in the cofinality of infinite cardinals.

(c) $\alpha > 0$ is cofinal in $\beta + \alpha$, in $\beta \cdot \alpha$, in β^α and in $\aleph_\alpha$.

(d) $n \in \omega$ is not cofinal in ω. Also, ω^2 is not cofinal in ω.

(e) $cf(\omega) = \omega$.

Theorem 52 *If α is cofinal in β and β is cofinal in γ, then α is cofinal in γ.*

Corollary 53 *If $\alpha = cf(\beta)$ and $\beta = cf(\gamma)$, then $\alpha \ge cf(\gamma)$.*

However this corollary can be improved to get $\alpha = cf(\gamma)$, which is the content of the next theorem:

Theorem 54 $cf(cf(\alpha)) = cf(\alpha)$.

Theorem 55 *Let $f : \alpha \to \beta$ have range cofinal in β. The $cf(\beta) \leq \alpha$.*

The previous theorem points to an alternative characterization of cofinality which could have been used in place of the definition above; namely, $cf(\beta)$ is equal to the least ordinal α for which there is a function $f : \alpha \to \beta$ which has range cofinal in β.

Definition A cardinal is *regular* iff $cf(\alpha) = \alpha$. A cardinal is *singular* if it is not regular.

Theorem 56 *For all $\kappa \geq \omega$, $cf(\kappa)$ is regular.*

Theorem 57 (AC) *Every successor cardinal, κ^+, is regular.*

Theorem 58 *If κ is singular, then κ can be written as a union of few than κ sets of size strictly less than κ, i.e.,*

$$\kappa = \bigcup \{x_i : i \in I\}$$

for some set I with $|I| < \kappa$ such that $|x_i| < \kappa$ for all $i \in I$.

Theorem 59 (AC) *κ^+ is regular.*

Assuming the Axiom of Choice, the converse to Theorem 58 also holds (see the homework). From examining these proofs, we see that $cf(\kappa)$ is equal to the least λ such that κ can be written as the union of λ many sets each of size strictly less than κ.

Lemma 60 *If α is cofinal in β, then $cf(\alpha) = cf(\beta)$.*

Theorem 61 *Let α be a limit ordinal. Then $cf(\omega_\alpha) = cf(\alpha)$.*

8.5 Inaccessible Cardinals

Definition A *strong limit* cardinal is an infinite cardinal κ such that $\kappa > 2^\lambda$ for all $\lambda < \kappa$.

Note that every strong limit cardinal is a limit cardinal.

Definition A *weakly inaccessible* cardinal is an uncountable, regular, limit cardinal. A *strongly inaccessible* (or just *inaccessible*) cardinal is an uncountable, regular, strong limit cardinal.

Theorem 62 *There exists (arbitrarily large) ordinals α such that $\aleph_\alpha = \alpha$.*

It is easy to see that a weakly inaccessible cardinal κ satisfies the property that $\aleph_\kappa = \kappa$; i.e., κ is the κ-th infinite cardinal. The inaccessible cardinals are just the cardinals which satisfy this property and which are regular.

The existence of (weakly) inaccessible cardinals cannot be proved from the axioms of $ZFC + GCH$. In fact, one cannot prove that ZF plus "there exists an inaccessible cardinal" is consistent, even from the assumption that ZF is consistent. (We'll prove this later in the course.)

8.6 Infinite Cardinal Sums and Products

We define infinite cardinal sums in terms of infinite disjoint union and infinite cardinal products in terms of infinite crossproduct.

Definition Let $\{\kappa_i : i \in I\}$ be a set of cardinals. Then, we define

$$\sum_{i\in I} = \left|\bigcup_{i\in I}(\{i\}\times\kappa_i)\right|.$$

$$\prod_{i\in I}\kappa_i = \left|\prod_{i\in I}\kappa_i\right| = |\{f : I \to \cup_i \kappa_i \text{ and } \forall i \in I, f(i) \in \kappa_i\}|$$

Remark: $\sum_{i\in I}\kappa = |I|\cdot\kappa$, and $\prod_{i\in I}\kappa = \kappa^{|I|}$. A homework problem has the complete characterization of the value of $\sum_{i\in I}\kappa_i$.

Theorem 63 König's Theorem *Let $I \neq \emptyset$, $\kappa_i < \lambda_i$ for all $i \in I$. Then*

$$\sum_{i\in I}\kappa_i < \prod_{i\in I}\lambda_i.$$

An easy corollary is something we've proved earlier:

Corollary 64 $\kappa < 2^\kappa$.

Corollary 65 $k < k^{cf(\kappa)}$.

Corollary 66 $cf(2^\kappa) > \kappa$.

We have proved three conditions about the function $\kappa \mapsto 2^\kappa$, for κ infinite:

$$\kappa \leq \lambda \;\Rightarrow\; 2^\kappa \leq 2^\lambda$$
$$\kappa < 2^{cf(\kappa)}$$
$$\kappa < cf(2^\kappa).$$

We shall see later on (Easton's theorem) that these are the *only* conditions that ZF can prove about this exponentiation function.

8.7 Exponentiation Assuming GCH

The following theorem completely determines cardinal exponentiation, under the assumption that the GCH is true.

Theorem 67 (AC+GCH) *Let $\kappa, \lambda \geq 2$, not both finite. Then,*
(1) *If $\kappa \leq \lambda$, then $\kappa^\lambda = \lambda^+$.*
(2) *If $\kappa > \lambda \geq cf(\kappa)$, then $\kappa^\lambda = \kappa^+$.*
(3) *If $\lambda < cf(\kappa)$, then $\kappa^\lambda = \kappa$.*

Definition We define $\beth_\alpha$ for all ordinals α, by letting $\beta_0 = \omega$, $\beth_{\alpha+1} = 2^{\beta_\alpha}$ and for limit ordinals β, letting $\beth_\beta = \sup\{\beth_\alpha : \alpha < \beta\}$.

GCH is equivalent to the assertion that $\beth_\alpha = \aleph_\alpha$ for all ordinals α.

9 Models of Set Theory

We start off with two examples of one element models for the language of set theory.

Example 1. Let $M = \{\emptyset\}$ and use the true $\in$ relation for M. This is a model of Extensionality, Union, Separation, Replacement, Foundation, and AC, but is not a model of Pairing, Power Set or Infinity. For Infinity, the successor function is not well-defined so you must be careful in interpreting the meaning of this axiom.

Example 2. Let M have a single element a (could be any fixed set) and in place of the membership relation use the binary relation E such that aEa.

This is a model of Extensionality, Pairing, Power Set, Replacement, Union, Choice, not not a model of Separation or Foundation. Again, the constant is not well-defined in this model, so it does not make sense to talk about the Infinity axiom being true or false in this model.

Example 3. Let $M = \mathbf{ON}$, i.e., the class of all ordinals, and use the true membership relation $\in$.

This is a model of Extensionality, Power Set, Union, Replacement, Foundation, and Infinity. It is not a model of Pairing or Separation. Note especially the surprise that this model satisfies the power set axiom!

Definition A *model* for the language of set theory consists of (a) a set or class M which is the collection of set in the universe of the model, and (b) a binary relation $E \supset M \times M$. If M is a set, we call the model a *set model*. Otherwise, we call M a *class model*.

An $\in$*-model* is a model in which E is the usual membership relation $\in$.

A *transitive model* is an $\in$-model which has a transitive universe M.

For general model theory, or logic, we usually consider arbitrary set models. For the special situation of set theory, we will usually consider $\in$-models, and these are almost always transitive.

9.1 Discussion

We had a discussion about "Formalism" and "Platonism". Formalists and Platonists have different views about the meaning of model-theoretic constructions and of relative consistency theorems.

Gödel's second incompleteness theorem precludes any reasonable model of set theory from proving its own consistency.

9.2 Relativization and the Definition of Truth

Definition Let φ be a first-order formula and M a model. The *relativization of* φ *to* M, denoted φ^M is defined inductively as follows:

MORE TO BE WRITTEN HERE

Note that φ^M has exactly the same free variables as φ, so if $\varphi = \varphi(x_1, \ldots, x_k)$, then we can also write $\varphi^M(x_1, \ldots, x_k)$.

Definition Let M be a model, let $\varphi(x_1, \ldots, m_k)$ be a first-order formula and let $m_1, \ldots,_k \in M$. We define "$\varphi(\vec{m})$ is true in M", or "M satisfies $\varphi(\vec{m})$, or "$M \vDash \varphi(\vec{m})$" to mean that $\varphi^M(m_1, \ldots, m_k)$ is true.

9.3 Absoluteness and Elementary Extensions

Definition of *absolute formula*, of *elementary equivalent* and of *elementary submodel*.

MORE TO BE WRITTEN HERE

9.4 Downward Löwenheim-Skolem Theorem

The following is a strong form of the Downward Löwenhiem Skolem Theorem.

Theorem 68 *Let N be an infinite model and let X be a subset of the universe of N. Then there is an elementary submodel M of N such that X is a subset of the universe of M and such that $|M| = \max\{\aleph_0, |X|\}$.*

If we apply the Downward Löwenhiem-Skolem theorem with N equal to the class of all sets, then we find that there is a countable $\in$-model of set theory. This is known as Skolem's paradox. In fact, we shall see that the Mostowski collapsing theorem can be used to further show that there exists a countable, transitive model of set theory.

Skolem's paradox. *There is a countable, transitive model of set theory.*

Discussion of why this is paradoxical. For example, any model of set theory must satisfy the formula which expresses the condition that there exists an infinite set.

Furthermore, the proof of Skolem's paradox cannot be carried out inside ZFC! The Downward Löwenheim-Skolem theorem needs the additional condition that N is a set model in order for the proof to be carries out in ZFC (as we shall see later).

10 First-order Logic

Definition of *language*, *constant symbol*, *predicate symbol*, *function symbol*, *logical symbols*, *term*, *formula*, *free* and *bound* variable occurrences and *term substitution*.

MORE TO BE WRITTEN HERE

Part II
Homework assignments

1. Assume y is a non-empty set. Prove that $\bigcap y$ exists.

2. The crucial property of ordered pairs is that they have unique first and second elements and are determined by these elements. Prove that $(x, y) = (u, v)$ iff $x = u$ and $y = v$.

3. Suppose someone has proposed

$$[x, y] \;:=\; \{x, \{x, y\}\}$$

as an alternative definition of ordered pair. Show that the property problem #2 hold for this definition too. (You may use the Axiom of Foundation for this problem.)

4. You need to use the Axiom of Foundation for this problem. Prove that $x \notin x$.

5. Let w be a set.

a. Let f be a set of order pairs which is a function. Write out carefully the definition of $z = f[w]$.

b. Let $\boldsymbol{F}$ be a function defined by a formula $\varphi(x, y)$. Write out carefully the definition of $\boldsymbol{F}[w]$.

6. Prove that AC is equivalent to AC'.

7. Prove that the well-ordering principle WO implies the Axiom of Choice AC.

8. (Harder) Prove that AC implies WO.

9. Suppose x and y are transitive. For each of the following assertions, either provide a proof or provide a counterexample.

a. $x \subset \wp(x)$.

b. $\wp(x)$ is transitive.

c. $x \cup y$, $x \cap y$, and $x \setminus y$ are transitive.

d. If $z \in x$, then z is transitive.

10. Prove the following, for all $x \in \omega$:

a. If $0 \neq x$, then $0 \in x$.

b. If $y \in x$, then $S(y) \in x$ or $S(y) = x$.

c. $x \subset \omega$.

11. Prove: For all x and y in ω, either $x \in y$, $x = y$ or $y \in x$.

12. Prove that there is no infinite sequence $(x_0, x_1, x_2, \ldots)$ such that $x_{i+1} \in x_i$ for all i. (Use the Axiom of Foundation.)

13. Write formulas with define the functions $x \cdot y$ and x^y over the natural numbers. Convince yourself of the existence and uniqueness conditions for your definitions; and that symbols for multiplication and exponentiation on the natural numbers can be used without exceeding the expressive power of the first-order language of set theory.

14. Do the same for the Fibanacci numbers defined by $F(0) = F(1) = 1$ and $F(n+2) = F(n) + F(n+1)$.

15. Prove that (the axioms of set theory imply that) integer addition is commutative and associative. (Hints are available.)

16. Prove that ordinal addition is associative: i.e., $\alpha+(\beta+\gamma) = (\alpha+\beta)+\gamma$.

17. Prove that the Hausdorff Maximal Principle is equivalent to the Axiom of Choice.

18. Prove that $\alpha \cdot \beta$ is the otype of the cross-product $\alpha \otimes \beta$.

19. Prove:

a. Each of $\alpha+\beta$, $\alpha \cdot \beta$ and α^β are strictly monotonically increasing as functions of β. (For multiplication, you must assume $\alpha > 0$, and for exponentiation, you must assume $\alpha > 1$.)

b. Each of $\alpha+\beta$, $\alpha \cdot \beta$ and α^β are non-decreasing as functions of α. For exponentiation, you must assume $\alpha > 0$.

20. Give a natural characterization (or "re-definition") of the order type α^β, similar in spirit to our characterizations of addition and multiplication in terms of concatenation and crossproduct.

21. Prove that if α and β are countable, then so are $\alpha+\beta$, $\alpha \cdot \beta$ and α^β. (There is no need to re-prove facts like the union of countably many sets is countable.)

22. Prove that ordinal arithmetic is left distributive. Give an example of how right distributivity can fail.

23. Prove the following about ordinal exponentiation.

a. $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$.

b. $(\alpha^\beta)^\gamma = \alpha^{\beta\cdot\gamma}$.

24. Prove the following:

a. For all α, $\omega^{\epsilon_\alpha} = \epsilon_\alpha$.

b. Prove that if $\omega^\beta = \beta$, then $\beta = \epsilon_\alpha$ for some α.

c. Prove that ϵ_α is the α-th fixed point of the map $\beta \mapsto \omega^\beta$. (Or at least make sense of this statement and convince yourself it is true.)

d. Prove that there is an ϵ-number α so that $\epsilon_\alpha = \alpha$.

25. (AC) Prove that a set is Dedekind finite iff it is finite.

26. Without using the Axiom of Choice, prove that for all cardinals κ, there is a cardinal $\lambda > \kappa$. [Hint: consider the supremum of the otypes of the well-orderings of κ.]

27. Suppose that $f : ON \to ON$ is a strictly increasing and continous function (i.e., $f(\sup A) = \sup f[A]$ for all sets of ordinals A.) Prove that there exist arbitrarily large fixed points of f. Prove that for all κ, there is an $\alpha > \kappa$ such that $\aleph_\alpha = \alpha$.

28. Give an example of cardinals α and β such that there is a function with domain α with range cofinal in b, and such that $cf(\beta) > cf(\alpha)$. Choose your example so that $\alpha < \beta$ if you are able.

29. Prove that there exists a strong limit ordinal.

30. Let κ be an infinite cardinal. Prove that $cf(\kappa)$ is equal to the smallest cardinal λ such that $\kappa = \bigcup \mathcal{I}$ for some set $\mathcal{I}$ of cardinality λ such that for all $X \in \mathcal{I}$, $|X| < \kappa$.

31. Prove that the Axiom of Choice is equivalent to Tukey's Lemma (see Kunen, problem #11, on page 44).

32. (Corrected from Kunen, problem #14, p. 44) Prove $|\{X \subset \kappa : |X| = \lambda\}|$ is equal to κ^λ.

33. Prove that the Pairing Axiom follows from the rest of the axioms of ZF^-.

34. Which axioms of set theory are true in the $\in$-model with universe ω (and the standard $\in$ relation).

35. (Kunen, #19, p. 45) Suppose that if κ is an infinite cardinal and $\lhd$ is a well-ordering of κ. Prove there is a subset X of κ such that $|X| = \kappa$, and such that $\lhd$ and $<$ (i.e., $\lhd$ and $\in$) agree on X.

36. a. Suppose κ is regular and $\lambda < \kappa$. Prove that κ^λ equals $\max\{\kappa, \sup\{\mu^\lambda : \mu < \kappa\}\}$.

b. Suppose κ is weakly inaccessible and $\lambda < \kappa$. Prove that κ^λ equals $\sup\{\mu^\lambda : \mu < \kappa\}$.

c. Suppose κ is inaccessible. Prove that $\kappa^{<\kappa} = \kappa$.

37. Suppose that κ is singular and is not a strong limit cardinal. Prove that $k^{<\kappa} = 2^{<\kappa} > \kappa$.

38. Suppose that κ is singular and is a strong limit cardinal. Prove that $2^{<\kappa} = \kappa$ and that $\kappa^{<\kappa} = \kappa^{cf(\kappa)} > \kappa$.

39. Prove that if M is a transitive model, then M satisfies Extensionality.

40. Prove that $\mathbf{HF} = V_\omega$.

41. What axioms of ZF are true in $V_{\omega+\omega}$?

42. Prove that the symmetry and transitivity of equality are provable in first-order logic. I.e., prove that

$$\vdash (\forall x)(\forall y)(x = y)$$

$$\vdash (\forall x)(\forall y)(\forall z)(x = y \wedge y = z \rightarrow x = z)$$

43. Let φ be any formula. Prove that $\vdash (\forall x)(\neg\varphi) \rightarrow \neg(\exists x)\varphi$, and that $\vdash (\exists y)[((\exists x)(\varphi(x))) \rightarrow \varphi(y)]$.

44. Fix an arbitrary language L.

a. Prove there is an infinite set T of sentences such that, for every L-structure $\mathcal{M}$, we have $\mathcal{M} \vDash L$ iff $|\mathcal{M}|$ is infinite.

b. Prove that there is no set T of sentences such that, for every L-structure $\mathcal{M}$, we have $\mathcal{M} \vDash L$ iff $|\mathcal{M}|$ is finite.

c. Prove that there is no finite set T of sentences such that, for every L-structure $\mathcal{M}$, we have $\mathcal{M} \vDash L$ iff $|\mathcal{M}|$ is infinite.

45. Suppose that $S \models T$ and $T \models S$ and that S is finite. Prove that there is a finite $R \subseteq T$ such that $R \models T$.

46. Let x, y be in **WF**.

a. Prove that $\cup x$ and $\wp(x)$ are in **WF**.

b. Prove that $\{x, y\}$, $x \times y$, ${}^y x$ are in **WF**.

c. Compute the ranks of these set in terms of the ranks of x and y.

d. Compute the ranks of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$.

47. Prove that $|V_\omega| = |\omega|$ without using the Axiom of Choice by giving an explicit isomorphism between the two sets. Hint: Given $n, m \in \omega$, let nEm hold iff the n-th bit of m's binary representation is equal to 1. Prove that $(V_\omega, \in)$ is isomorphic to (ω, E). (See problem III.5 in Kunen.)

48. Kunen, problem 15, page 108. Let AR be the strenthened Axiom of Replacement:

$$\forall x \in a \exists y \varphi \to \exists b \forall x \in a \exists y \in b \varphi.$$

Prove that AR is a theorem of ZF (with Foundation). See Kunen for a hint.

49. Kunen, problem 17, page 108. Prove in ZF, that if **R** is a well-founded relation on a class **A** (not necessarily set-like), then every non-empty subclass **X** of **A** has an **R**-minimal element. See Kunen for a hint!

50. (Kunen, page 147, #4.) Show that if $\kappa > \omega$, then $|H(\kappa)| = 2^{<\kappa}$.

51. (Kunen, page 147, #5.) Prove, for all $\kappa > \omega$, that $H(\kappa) = V_\kappa$ iff $\kappa = \beth_\kappa$.

52 . Let κ be strongly inaccessible. Prove that "α is strongly inaccessible" is absolute for $H(\kappa)$.

54. Kunen, page 147. #7.

55. Kunen, page 147. #8. Further hint, see Theorem I.7.6 on page 17.

56. Prove that the composition of absolute functions is absolute.

57. Prove that (ZFC proves that)

$$Con(ZFC^-) \quad \Rightarrow \quad Con(ZFC^- + (\exists x)(x = \{x\})).$$

Kunen's problems 18 and 19 on page 148 have hints for this.

58. Prove also that the sentence φ in Gödel's incompleteness theorem can be picked so that both φ and $\neg\varphi$ unprovable. [Hint: look up Rosser's theorem.]

59. (Kunen, #2, page 180.) (AC) Let $\alpha \geq \omega$. Show that $|L(\alpha)| = |V_\alpha|$ if and only if $\alpha = \beth_\alpha$.

60. Kunen, #3, page 180. Assume $V = L$. Show that if $\alpha > \omega$, then $L(\alpha) = V_\alpha$ if and only if $\alpha = \beth_\alpha$.

61. Kunen, #4, page 180. Assume $V = L$. Show that $L(\kappa) = H(\kappa)$ for all infinite cardinals κ.

62. Kunen, #6, page 180. Definition and properties of $L(A)$.

63. Kunen, #7, page 180.