# Math 268: Seminar in Combinatorics - Logic and Randomness

## 1. Motivation and introduction

The unpredictability perspective on randomness suggests that even if we have full information about finite pieces of the object, we are no more likely to predict subsequent bits. This can be formalized using **martingales**. Essentially, the idea is that there is no effective betting strategy that will have unbounded success trying to bet on successive observations.

## 2. Definitions

**Definition.** A (monotonic) **stake function** is a function $q : \{0,1\}^* \to [0,2]$. We interpret the function as saying that

- if $q(\sigma) < 1$, bet that $X(|\sigma|) = 1$ and
- if $q(\sigma) > 1$, bet that $X(|\sigma|) = 0$ and
- if $q(\sigma) = 1$, stay agnostic about value of $X(|\sigma|)$.

Thus, we define the **payoff function** for nonempty finite strings as $c_q : \{0,1\}^* \to [0,2]$ as

$$c_q(\sigma 0) = q(\sigma) \qquad c_q(\sigma 1) = 2 - q(\sigma)$$

Then, the **capital function** is defined $d_q : \{0,1\}^* \to \mathbb{R}^{\geq 0}$ as

$$d_q(\sigma_0\sigma_1 \cdots \sigma_n) = c_q(\sigma_0) \cdot c_q(\sigma_0\sigma_1) \cdots c_q(\sigma_0\sigma_1 \cdots \sigma_n)$$

(by convention set initial capital to 1, i.e. $d_q(\epsilon) = 1$).

The notion of martingales from probability theory can be stated as a special case to mean:

**Definition.** A function $d : \{0,1\}^* \to \mathbb{R}^{\geq 0}$ is a **martingale** if for all $\sigma \in 2^{<\omega}$,

$$d(\sigma) = \frac{d(\sigma 0) + d(\sigma 1)}{2}.$$

Essentially, this corresponds to a fair game: the expected value of capital after the next bid is the same as the current amount.

**Fact.** *For any stake function, $q$, the capital function $d_q$ is a martingale.*

*Proof.* By definition,

$$\begin{aligned}\frac{d_q(\sigma 0) + d_q(\sigma 1)}{2} &= \frac{d_q(\sigma)c_q(\sigma 0) + d_q(\sigma)c_q(\sigma 1)}{2} = d_q(\sigma)\frac{c_q(\sigma 0) + c_q(\sigma 1)}{2} \\ &= d_q(\sigma)\frac{q(\sigma) + 2 - q(\sigma)}{2} = d_q(\sigma)\end{aligned}$$

□

**Fact.** *Any martingale $d : \{0,1\}^* \to \mathbb{R}^{\geq 0}$ determines a stake function.*

*Proof.* For consistency with the convention that initial capital is 1, assume that $d(\epsilon) = 1$. Then the stake function associated with $d$ is defined by

$$q_d(\sigma) = \frac{d(\sigma 0)}{d(\sigma)}.$$

Notice that by the martingale property,

$$q_d(\sigma) = \frac{d(\sigma 0)}{d(\sigma)} = 2 - \frac{d(\sigma 1)}{d(\sigma)}$$

We prove by induction that the capital function associated with $q_d$ is exactly $d$. The initial capital is 1 by assumption. It remains to show that for any $\sigma \in \{0,1\}^*$, $d(\sigma 0) = c_{q_d}(\sigma_0)\cdots c_{q_d}(\sigma)c_{q_d}(\sigma 0)$ and $d(\sigma 1) = c_{q_d}(\sigma_0)\cdots c_{q_d}(\sigma)c_{q_d}(\sigma 1)$. By induction, we assume that

$$d(\sigma) = c_{q_d}(\sigma_0)\cdots c_{q_d}(\sigma).$$

By definition of the payoff function, $c_{q_d}(\sigma 0) = q_d(\sigma)$ and $c_{q_d}(\sigma 1) = 2 - q_d(\sigma)$. Thus,

$$c_{q_d}(\sigma_0)\cdots c_{q_d}(\sigma)c_{q_d}(\sigma 0) = d(\sigma)c_{q_d}(\sigma 0) = d(\sigma)q_d(\sigma) = d(\sigma)\frac{d(\sigma 0)}{d(\sigma)} = d(\sigma 0)$$

as required, and

$$c_{q_d}(\sigma_0)\cdots c_{q_d}(\sigma)c_{q_d}(\sigma 1) = d(\sigma)c_{q_d}(\sigma 1) = d(\sigma)(2-q_d(\sigma)) = d(\sigma)\left(2 - \left(2 - \frac{d(\sigma 1)}{d(\sigma)}\right)\right) = d(\sigma 1).$$

□

From the definitions we have the reductions, $d_q \leq_T q$ and $q_d \leq_T d$. Thus, we equate the two notions of **martingale** and **monotonic capital function**. (This will be useful when generalizing to nonmonotonic functions.) To extend these notions to infinite strings, it will be useful to switch notation slightly.

**Definition.** Let $X \in \{0,1\}^\omega$ (an infinite string) and fix a stake function $q : \{0,1\}^* \to [0,2]$. Define the **payoff function while playing on** $X$ to be $c_q^X : \mathbb{Z}^+ \to [0,2]$ given by

$$c_q^X(n+1) = \begin{cases} q(X \upharpoonright n+1) & \text{if } X(n+1) = 0 \\ 2 - q(X \upharpoonright n+1) & \text{if } X(n+1) = 1 \end{cases}$$

and the **capital function while playing on** $X$ to be $d_q^X : \mathbb{Z}^+ \to \mathbb{R}^{\geq 0}$ given by

$$d_q^X(n) = \Pi_{i=1}^n c_q^X(i)$$

**Definition.** A martingale $d$ **succeeds** on a sequence $X$ if

$$\limsup_{n\to\infty} d^X(n) = \infty.$$

Recall: The limit superior of $x_n$ is the supremum of subsequential limits. And,

$$\limsup_n x_n = C \iff \forall b > C \exists N \forall n \geq N (x_n < b)$$

$$\limsup_n x_n = \infty \iff \forall k \exists n x_n \geq k$$

**Fact.** *The definition above is consistent with the interpretation of the stake function*

*Proof.* The stake function will succeed on sequence $X$ only if

$$\exists^\infty i \left(c_b^X(i) > 1\right).$$

But, $c_b^X(i) > 1$ if and only if

$q(X \upharpoonright i) > 1$ and $X(i) = 0$ (in which case the payoff is $q(X \upharpoonright i)$ ), or

$q(X \upharpoonright i) < 1$ and $X(i) = 1$, (so the payoff is $2 - q(X \upharpoonright i) > 2 - 1 > 1$.

That is, only if we infinitely often bet correctly. □

**Fact.** *For each $X \in 2^\omega$, there is a martingale that succeeds on it.*

*Proof.* Define the martingale that exactly follows $X$ by the stake function

$$q(X \upharpoonright n) = 2(1 - X(n))$$

and for $\sigma \not\prec X$, put $q(\sigma) = 1$. Then if $X(n) = 0$, $c_q^X(n) = q(X \upharpoonright n) = 2$; and if $X(n) = 1$, $c_q^X(n) = 2 - q(X \upharpoonright n) = 2(1 - 0) = 2$. Therefore,

$$d^X(n) = \Pi_{i=1}^n c_q^X(i) = \Pi_{i=1}^n 2 = 2^n$$

and

$$\limsup_{n\to\infty} d^X(n) = \infty.$$

□

But, this function may be very complicated to compute. In fact, as complicated as $X$.

Intuition: **the minimal complexity of a martingale which succeeds on $X$ tells us something about the randomness of $X$.**

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**Definition** (DH p. 202)**.** Let $f : \{0,1\}^* \to \mathbb{R}^{\geq}$. The corresponding set of left cuts is

$$L_d = \{(\sigma, r) : \sigma \in \{0,1\}^*, r \in \mathbb{Q}^{\geq}, r < f(\sigma)\}.$$

The function $f$ is **computable** if the set above is uniformly computable. The function is **computably enumerable** if the set above is uniformly c.e.

**Definition** (DH p. 236, 270, 303)**.** A sequence is **computably random** if there is no (total) computable martingale that succeeds on it.

A sequence is **partial computably random** if there is no partial computable martingale that succeeds on it. Partial computable martingales are partial computable functions $d : \{0,1\}^* \to \mathbb{R}^{\geq 0}$ such that,

- if $d(\sigma i)$ is defined then so are $d(\sigma)$ and $d(\sigma(1-i))$;
- where defined, $d$ satisfies martingale property; and
- succeeds on $X$ if and only if $\limsup_n d(X \upharpoonright n) = \infty$ and $d(X \upharpoonright n) \downarrow$ for all $n$.

A sequence is **ML-random** if there is no computably enumerable martingale that succeeds on it.

**Fact** (DH p. 270, 303)**.** *For each (partial) computable martingale* $d : \{0,1\}^* \to \mathbb{R}^{\geq}$*, there is a (partial) computable martingale* $\tilde{d} : \{0,1\}^* \to \mathbb{Q}^{\geq}$ *with the same success sets. In fact,* $d(\sigma) < \tilde{d}(\sigma) < d(\sigma) + 1$ *for each* $\sigma$*.*

*Proof.* We proceed by induction. By convention, $d(\epsilon) = \tilde{d}(\epsilon) = 1$. Suppose $\tilde{d}(\sigma) \in \mathbb{Q}^{\geq}$ and $d(\sigma) < \tilde{d}(\sigma) < d(\sigma) + 1$. We define a procedure such that if $d(\sigma 0) \uparrow$ then $\tilde{d}(\sigma i) \uparrow$ as well. Otherwise, each question "$(\sigma 0, r) \in L_d$?" is answered since $L_d$ is computable. Hence, enumerate pairs of non-negative rationals $(x_1, y_1), (x_2, y_2), \ldots$ and ask whether

- $x_j + y_j = 2\tilde{d}(\sigma) - 1$;
- $(\sigma 0, x_j) \in L$ and $(\sigma 1, y_j) \in L$;
- $(\sigma 0, x_j + \frac{1}{2}) \notin L$ and $(\sigma 1, y_j) \notin L$.

Such a $j$ exists (if $d(\sigma 0) \downarrow$) since $d$ satisfies the martingale condition and $L_d$ contains the left cuts of $d(\sigma 0)$ and $d(\sigma 1)$. Given such a $j$ then

$$x_j < d(\sigma 0) \leq x_j + \frac{1}{2} \qquad \text{and} \qquad y_j < d(\sigma 1) \leq y_j + \frac{1}{2}$$

and adding 1 to both sides of the first inequalities we get

$$x_j + \frac{1}{2} < x_j + 1 < d(\sigma 0) + 1 \qquad \text{and} \qquad y_j + \frac{1}{2} < y_j + 1 < d(\sigma 1) + 1$$

Thus,

$$d(\sigma 0) < x_j + \frac{1}{2} < d(\sigma 0) + 1 \qquad \text{and} \qquad d(\sigma 1) < y_j + \frac{1}{2} < d(\sigma 1) + 1.$$

Put $\tilde{d}(\sigma 0) = x_j + \frac{1}{2}$ and $\tilde{d}(\sigma 1) = y_j + \frac{1}{2}$. By the first condition on $(x_j, y_j)$, the martingale condition is satisfied. By the inequalities above, the error on $\tilde{d}$ remains bounded.

□

Note: the above proof used tests for *non-membership* as well as membership in $L$ so it can't be transferred to c.e. martingales.

**Fact** (DH p. 275, 304)**.**

*ML randoms* $\subsetneq$ *partial computable randoms* $\subsetneq$ *computable randoms.*

*Proof.* To prove that ML randoms are partial computable randoms, need that if a sequence is not p.c. random then it is not ML random. It is sufficient to show that any p.c. martingale can be transformed into a c.e. one which succeeds on the same strings. Let $d$ be a p.c. martingale. Define $\hat{d}(\sigma)$ to be zero while $d(\sigma)$ is undefined and, if the computation of $d(\sigma)$ ever converges, put $\hat{d}(\sigma) = d(\sigma)$. □

Is there a way to capture the notion of *effective* martingales that leads to an equally strong notion of randomness as ML randomness? In particular, we look at **non-monotonic betting**. As before, we use stake functions and their associated payoff and capital functions. But, we must also specify a scan function which chooses which bit to bet on next.

**Definition.** A **finite assignment** is a sequence $x = (n_i, a_i) \in (\mathbb{N} \times \{0,1\})^*$. We interpret a finite assignment as listing the place and value of bits scanned so far. That is, for a given string $X$, a finite assignment would say that

$$X(n_i) = a_i$$

for each $i = 0, \ldots, n-1$. We denote by $FA$ the set of all finite assignments; that is, $FA = (\mathbb{N} \times \{0,1\})^*$.

A **scan rule** is a partial function $s : FA \to \mathbb{N}$ such that

$$\forall w \in FA\, (s(w) \notin \mathrm{dom}(w))\,.$$

In other words, any partial function that does not repeatedly look at any bits in the string.

A **nonmonotonic stake function** is a partial function $q : FA \to [0,2]$.

A **nonmonotonic betting strategy** is a pair $b = (s,q)$ where $s$ is a scan rule and $q$ is a nonmonotonic stake function. Alternatively, a nonmonotonic betting strategy is $b : \{0,1\}^* \to \mathbb{N} \times [0,2]$ and we interpret

$$b(\sigma) = (\text{next bit to scan}, \text{amount to bet})$$

where $\sigma$ is the sequence of bit-values in the locations we chose to scan (and the "next bit to scan" always satisfies the "no repeated scan" rule). Note that given such a function, the associated scan and stake rules can be recovered inductively.

**Definition.** Fix a nonmonotonic betting strategy $b = (s,q)$ and an infinite string $X$.

The **observation function** is a partial function $o_b^X : \mathbb{N} \to FA$ defined by

$$o_b^X(0) = \epsilon \qquad \text{and} \qquad o_b^X(k+1) = o_b^X(k)\,\hat{}\,\big(s(o_b^X(k)), X(s(o_b^X(k)))\big)$$

if $o_b^X(k), s(o_b^X(k))$ are both defined (and it is undefined otherwise).

Then the **payoff function** (as before) is given by a partial function $c_b^X : \mathbb{N}^+ \to [0,2]$ where

$$c_b^X(k+1) = \begin{cases} q(o_b^X(k)) & \text{if } X(s(o_b^X(k))) = 0 \\ 2 - q(o_b^X(k)) & \text{if } X(s(o_b^X(k))) = 1 \end{cases}$$

and, the **capital function** is defined inductively by $d_b^X(0) = 1$ (initial capital always set to 1), and

$$d_b^X(k) = \Pi_{i=1}^k c_b^X(i)$$

We say that the nonmonotonic betting strategy $b$ **succeeds** on $X$ if

$$\limsup_{k\to\infty} d_b^X(k) = \infty.$$

**Definition.** A sequence is **KL-random** if there is no computable nonmonotonic betting strategy that succeeds on it.

**Fact** (Merkle 2003; DH p. 310)**.** *A sequence is KL random if and only if no partial computable betting strategy succeeds on it.*

*Proof.* Given a partial computable nonmonotonic betting strategy that succeeds on some string $X$, build a total computable nonmonotonic betting stratgey that also succeeds on $X$. Let $s_0, s_1, \ldots$ be the sequence of locations of $X$ scanned by the partial computable strategy $d$ such that

$$\limsup_k d(X(s_k)) = \infty.$$

Partition $\{s_i\}$ into odd and even locations; it must be the case that $\limsup = \infty$ for at least one of the corresponding subsequences. Define a total computable nonmonotonic betting strategy that emulates $d$ on the subsequence with $\limsup = \infty$ and uses the other subsequence to "wait" for convergence of each of the computations. □

**Fact.**

*KL randoms $\subsetneq$ partial computable randoms $\subsetneq$ computable randoms.*

*Proof.* Subset inclusions go by definitions:

$$\{\text{nonmonotonic partial computable strategy}\} \supseteq \{\text{monotonic partial computable strategy}\}$$

□

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How do nonmonotonic betting strategies relate to martingales? Let $b = (s, q)$. Recast the payoff and capital functions as functions of the observations thus far (a finite assignment). First, $c_b : FA \to [0, 2]$ is defined inductively by $c_b(\epsilon) = 1$ and for $w \in FA$

$$c_b(w\,\hat{}\,(s(w), 0)) = q(w) \qquad\qquad c_b(w\,\hat{}\,(s(w), 1)) = 2 - q(w).$$

That is, if the next bit we scan is a 0, the payoff is $q(w)$ and if the bit is 1, the payoff is $2 - q(w)$. Notice that $c_b$ is a partial function: it is undefined for each FA not consistent with the scan function of $b$.

Now, $d_b : FA \to [0, 2]$ is $d_b(\epsilon) = 1$ and for $w \in FA$ and $i \in \{0, 1\}$

$$d_b(w\,\hat{}\,(s(w), i)) = d_b(w)c_b(w\,\hat{}\,(s(w), i)).$$

*Check that this agrees with previous definitions for any fixed infinite string $X$:*

**Fact.** *For any $X \in \{0, 1\}^\omega$ and for all $k \in \mathbb{N}$*

$$d_b^X(k) = d_b(o_b^X(k)).$$

*That is, the first definition we gave for $d_b$ (LHS) agrees with the second (RHS).*

*Proof.* By induction on $k$. If $k = 1$ then

$$\begin{aligned} d_b^X(1) = c_b^X(1) &= \begin{cases} q(o_b^X(0)) & \text{if } X(s(o_b^X(0))) = 0 \\ 2 - q(o_b^X(0)) & \text{if } X(s(o_b^X(0))) = 1 \end{cases} \\ &= \begin{cases} q(\epsilon) & \text{if } X(s(\epsilon)) = 0 \\ 2 - q(\epsilon) & \text{if } X(s(\epsilon)) = 1 \end{cases} \\ &= c_b\left[\epsilon\,\hat{}\,\big(s(\epsilon), X\big(s(\epsilon)\big)\big)\right] = c_b(o_b^X(1)) = d_b(o_b^X(1)) \end{aligned}$$

Similarly, if true for $k$ then

$$\begin{aligned} d_b^X(k+1) = d_b^X(k)c_b^X(k+1) &= \begin{cases} d_b^X(k)q(o_b^X(k)) & \text{if } X(s(o_b^X(k))) = 0 \\ d_b^X(k)(2-q(o_b^X(k))) & \text{if } X(s(o_b^X(k))) = 1 \end{cases} \\ &\overset{\text{IH}}{=} \begin{cases} d_b(o_b^X(k))q(o_b^X(k)) & \text{if } X(s(o_b^X(k))) = 0 \\ d_b(o_b^X(k))(2-q(o_b^X(k))) & \text{if } X(s(o_b^X(k))) = 1 \end{cases} \\ &= d_b(o_b^X(k))c_b(o_b^X(k+1)) = d_b(o_b^X(k+1)). \end{aligned}$$

□

The martingale property can now be phrased as: a function $d : FA \to \mathbb{R}^{\geq}$ is a **nonmonotonic martingale** if for any $w \in FA$, there is an $n \in \mathbb{N}$

$$d(w) = \frac{d(w\,\hat{}\,(n,0)) + d(w\,\hat{}\,(n,1))}{2}$$

The capital functions of nonmonotonic betting strategies are nonmonotonic martingales: For $b = (s,q)$ let $w \in FA$ and put $n = s(w)$. Then

$$\begin{aligned} \frac{d_b(w\,\hat{}\,(n,0)) + d_b(w\,\hat{}\,(n,1))}{2} &= \frac{d_b(w)c_b(w\,\hat{}\,(n,0)) + d_b(w)c_b(w\,\hat{}\,(n,1))}{2} \\ &= \frac{d_b(w)q(w)) + d_b(w)(2-q(w))}{2} = d_b(w) \end{aligned}$$

**Definition.** For $b = (q,s)$ a nonmonotonic betting strategy and $w \in FA$, we define what it means for a finite assignment $v \in FA$ extending $w$ to be **consistent with** $b$ by induction on $|v| - |w|$:

- if $|v| - |w| = 1$ then $v = w\,\hat{}\,(n,i)$ for some $n \in \mathbb{N}, i \in \{0,1\}$. In this case: $v$ is consistent with $b$ if and only if $n = s(w)$.
- if $|v| - |w| = k+1$ and $v = w\,\hat{}\,u\,\hat{}\,(n,i)$ is such that $w\,\hat{}\,u$ is consistent with $b$ then $v$ is consistent with $b$ if and only if $n = s(w\,\hat{}\,u)$.

**Lemma** (Kolmogorov's Inequality extended to nonmonotonic strategies)**.** *If $b = (q,s)$ is a nonmonotonic betting strategy, $w \in FA$, $S$ a prefix-free set of extensions of $w$ consistent with $b$ then*

$$\sum_{v \in S} 2^{-|v|} d_b(v) \leq 2^{-|w|} d_b(w).$$

*Proof.* We first prove by induction that the inequality holds for all finite $S$. If $|S| = 1$, then $w \preceq v$ and let $m = |v| - |w|$ so write

$$v = w\,\hat{}\,(n_1,i_1)\,\hat{}\,\cdots\,\hat{}\,(n_m,i_m).$$

Moreover, since $v$ is an extension of $w$ consistent with $b$, each $n_k = s(w\,\hat{}\,n_1\,\hat{}\,\cdots\,\hat{}\,n_{k-1})$. By the nonmonotonic martingale property (with $n_1$ as a valid witness)

$$d_b(w\,\hat{}\,(n_1,i_1)) \leq d_b(w\,\hat{}\,(n_1,0)) + d_b(w\,\hat{}\,(n_1,1)) = 2d_b(w)$$

Repeating the above $m-1$ more times, we get

$$d_b(v) \leq 2^m d_b(w).$$

Intuitively, this corresponds to the fact that each bet at most doubles the capital.

Suppose the inequality holds for all sets of size $k$ and consider $S$ with $|S| = k+1$. Let $u =$ longest extension of $w$ consistent with $b$ such that for all $v \in S$, $v$ is an extension of $u$ consistent with $b$. Define

$$S_0 = \{v \in S : u\hat{\ }(s(u), 0) \preceq v\} \qquad S_1 = \{v \in S : u\hat{\ }(s(u), 1) \preceq v\}.$$

These are disjoint subsets of $S$ and each is nonempty (by prefix-free assumption and consistency with $b$) so the inductive hypothesis applies to each one. Computing,

$$\begin{aligned}
\sum_{v \in S} 2^{-|v|} d_b(v) &= \sum_{v \in S_0} 2^{-|v|} d_b(v) + \sum_{v \in S_1} 2^{-|v|} d_b(v) \\
&\overset{\text{IH}}{\leq} 2^{-|u|+1} d_b(u0) + 2^{-|u|+1} d_b(u1) = 2^{-(|u|+1)}\big(d_b(u0) + d_b(u1)\big) \\
&\overset{\text{nm martingale}}{=} 2^{-(|u|+1)}(2d(u)) = 2^{-|u|} d(u) \\
&\overset{\text{case } n=1}{\leq} 2^{-|w|} d(w).
\end{aligned}$$

Thus, the induction is complete. If $S$ is infinite, let $v_1, v_2, \ldots$ be an ordering of its elements. By definition of infinite sum,

$$\sum_{v \in S} 2^{-|v|} d_b(v) = \lim_{N \to \infty} \sum_{i=1}^{N} 2^{-|v_i|} d_b(v_i)$$

Each of the partial sums is bounded by $2^{-|w|} d_b(w)$ (since the induction above works for each partial sum), hence the limit also satisfies the inequality. □

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**Fact** (Muchnik, Semenov, Uspensky; DH p. 311)**.** *If a sequence is ML random then it is KL random.*

*Proof.* We will show that if a sequence fails to be KL random then it is not ML random either. The criteria for non-ML-randomness will be in terms of tests.

**Definition.** A **Martin-Löf test** (ML test) is a uniformly c.e. sequence of open sets $\{V_k\}$ such that

$$\forall k \lambda(V_k) \leq 2^{-k}$$

(where $\lambda$ is Lebesgue measure). An infinite string $X \in \{0,1\}^\omega$ is disqualified by the ML test if

$$X \in \bigcap_k V_k.$$

What does it mean for a sequence of open sets to be uniformly c.e.? Recall that an open set is a countable union of basic open sets, which are cones above finite strings. We define an open set to be c.e. if and only if the set of representatives of its basic open sets is c.e. The intuition is that an enumeration of these representatives will tell us in finite time that a given infinite string is in the set (if it is).

**Fact.** *An infinite string $X \in \{0,1\}^\omega$ is ML random if and only if it passes all ML tests. That is, no computably enumerable monotonic betting strategy succeeds on $X$ if and only if $X$ passes all ML tests.*

Back to proof of Muchnik, Semenov, Uspensky: Fix $X \in \{0,1\}^\omega$ and $b = (q,s)$ such that $\limsup_n d_b^X(n) = \infty$. Assume (wlog) that $s$ is a total function and hence so is $d_b$. Define sequence of sets $V_k$ by, for each $w \in FA$,

$$[w] \subseteq V_k \iff d_b(w) \geq 2^k \text{ and } w \text{ is consistent with } b$$

where

$$[w] = \{\text{all infinite strings that agree with } w\} = \{Y \in \{0,1\}^\omega : \forall 0 \leq j < |w| \, (Y((w_j)_1) = (w_j)_2)\},$$

a finite union of basic open sets (cones). Notice that for each $Z \in \{0,1\}^\omega$, for each $n \in \mathbb{N}$,

$$Z \in [o_b^Z(n)].$$

We will show that $\{V_k\}$ is a ML test. It is uniformly c.e. because $d_b(w)$ is approximable so when we see $(w,r) \in L_{d_b}$ for some $r \geq 2^k$ we can conclude that

$$2^k \leq r < d_b(w)$$

so $w \in V_k$. Then, enumerate $w$ into $V_k$. It remains to show that the measure of $V_k$ is no bigger than $2^{-k}$.

**Fact.** *For each $w \in FA$, $\lambda([w]) = 2^{-|w|}$.*

*Proof.* Let $m = \max \operatorname{dom}(w)$. Let $|w| = \ell$. Then there are $m - \ell$ undetermined bits in the $m$-long prefix of any infinite string extending $w$. For $0 \leq r < 2^{m-\ell}$, define $\sigma_r$ as the $m$-long binary string which agrees with $w$ on bit locations in the domain of $w$ and which fills in the rest of the bits with the $r$th binary string of length $m - \ell$. Then

$$[w] = \bigcup_{0 \leq r < 2^{m-\ell}} [\sigma_r]$$

and this is a disjoint union. Therefore,

$$\lambda([w]) = \sum_{0 \leq r < 2^{m-\ell}} \lambda([\sigma_r]) = \sum_{0 \leq r < 2^{m-\ell}} 2^{-m} = 2^{-m} 2^{m-\ell} = 2^{-\ell} = 2^{-|w|}.$$

□

Let $B_k$ be a c.e. prefix-free subset of $\{w : d_b(w) \geq 2^k\}$ consistent with $b$ and such that $[B_k] = V_k$. Then

$$\lambda(V_k) = \lambda([B_k]) = \sum_{w \in B_k} 2^{-|w|}.$$

For each $w \in B_k$, $d_b(w) \geq 2^k$ so $\frac{d_b(w)}{2^k} \geq 1$ and using Kolmogorov's Inequality for nonmonotonic betting strategies,

$$\lambda(V_k) = \sum_{w \in B_k} 2^{-|w|} \leq \sum_{w \in B_k} 2^{-|w|} \frac{d_b(w)}{2^k} = 2^{-k} \sum_{w \in B_k} 2^{-|w|} d_b(w) \overset{\text{by K.I}}{\leq} 2^{-k} 2^{|\epsilon|} d_b(\epsilon) = 2^{-k}.$$

Thus, $\{V_k\}$ is a ML test. Moreover, for any $X \in 2^\omega$,

$$\begin{aligned} d_b \text{ succeeds on } X &\iff \limsup_n d_b^X(n) = \infty \\ &\iff \forall C \exists n \ \left(d_b^X(n) \geq C\right) \\ &\iff \forall k \exists n \ \left(d_b(o_b^X(n)) \geq 2^k\right) \\ &\iff X \in \cap_k V_k \iff X \text{ fails } \{V_k\}. \end{aligned}$$

*This is basically the same proof as that showing computably enumerable martingales gives ML tests.*

□

**Question.** Is each KL random also ML random?

## 3. Exploring nonmonotonic randomness

**Theorem** (DH p. 313)**.** *No partial computable nonmonotonic betting strategy succeeds on all computably enumerable sequences.*

*Proof.* Given a betting strategy $b = (q, s)$, we will define a c.e. set $W$ such that each bet against $W$ is lost. To do so, we keep track of which bets $b$ would make and ensure the bet is lost. That is, define the sequence of finite assignments

$$x_0 = \epsilon \qquad x_{k+1} = x_n \hat{\ } (s(x_k), a_{k+1})$$

where

$$a_{k+1} = \begin{cases} 1 & \text{if } q(x_k) \geq 1 \\ 0 & \text{if } q(x_k) < 1. \end{cases}$$

In order for this sequence of finite assignments to reflect a game on $W$, enumerate $n_k$ into $W$ if $(n_k, 1)$ is in some $x_i$. Notice that this is a computable enumeration if $b$ is (partial) computable: the sequence of finite assignments can be built computably in $b = (q, s)$ since we wait until an appropriate lower/ upper bound appears for $q(x_k)$ to guarantee the order relationship between $q(x_k)$ and 1. Once we know which case we're in, $a_{k+1}$ is fixed and $n_{k+1}$ is given by $s(x_k)$. This enumeration of the sequence of finite assignments, in turn, yields the enumeration of $W$ since with each new $x_k$ we enumerate at most one new element into $W$.

Since $b$ may be partial, $s(x_k)$ may be undefined from some point on, in which case $W$ will be finite. In this case, $b$ loses against $W$ because it doesn't have "time" to amass unbounded capital.

Assume, on the other hand, that $W$ is infinite and so is the sequence of finite assignments $\{x_k\}$. Then $b$ loses every bet against $W$. That is,

$$\begin{aligned} d_b^W(k+1) = d_b^W(k) c_b^W(k+1) &= \begin{cases} d_b^W(k) q_b(o_b^W(k)) & \text{if } W(s(o_b^W(k))) = 0 \\ d_b^W(k)(2 - q_b(o_b^W(k))) & \text{if } W(s(o_b^W(k))) = 1 \end{cases} \\ &= \begin{cases} d_b^W(k) q_b(o_b^W(k)) & \text{if } q_b(o_b^W(k)) < 1 \\ d_b^W(k)(2 - q_b(o_b^W(k))) & \text{if } q_b(o_b^W(k)) \geq 1 \end{cases} \leq d_b^W(k). \end{aligned}$$

Thus the capital is a nonincreasing function and $b$ loses on $W$. □

**Corollary.** *There is no universal nonmonotonic test for randomness.*

On the other hand, there is a universal ML test for randomness (Martin-Löf, 1966)

**Fact** (DH p. 313)**.** *Two computable nonmonotonic betting strategies are enough to succeed on all c.e. sets.*

*Proof.* Intuitively: The first betting strategy will be to always bet the next bit is 0. This will succed on each infinite string with enough zeroes (where "enough" can be quantified by the proportion we stake on each bet). The second betting strategy will exploit the fact that the proportion of zeroes is lower than this threshold.

Define $b_0 = (s_0, q_0)$ by

$$s_0(\epsilon) = 0 \qquad s_0(w) = 1 + \max \mathrm{dom} w$$

for each $w \in FA \setminus \{\epsilon\}$. Also, define $q_0(w) = \frac{5}{3}$ for all $w \in FA$.

□

Conclusion: KL-nullsets behave differently from ML-nullsets.

## 4. ML-random inside KL-randoms

Definition of splittings.

**Theorem** (van Lambalgen's Lemma)**.** *Let $Z$ be computable, infinite, co-infinite subset of $\mathbb{N}$. Then $A = A_0 \oplus_Z A_1$ is ML random if and only if*

- *$A_0$ is ML random, and*
- *$A_1$ is ML random relative to $A_0$.*

**Question** (Jason Teutsch)**.** Suppose $Z$ is c.e. infinite and co-infinite subset of $\mathbb{N}$ and $A = A_0 \oplus_Z A_1$ is ML random. Does this imply that $A_0$ is ML random as well?

The current analogue for KL randomness is a little weaker.

**Theorem** (DH p. 314)**.** *Let $Z$ be a computable, infinite, co-infinite subset of $\mathbb{N}$. Then $A = A_0 \oplus_Z A_1$ is KL-random if and only if*

- *$A_0$ is KL random* $\boxed{\textit{relative to } A_1}$*, and*
- *$A_1$ is KL random relative to $A_0$.*

This theorem means that we can find ML randoms inside KL randoms.

**Fact.** *Let $Z$ be a computable, infinite, co-infinite subset of $\mathbb{N}$. If $A = A_0 \oplus_Z A_1$ is KL-random then at least one of $A_0, A_1$ is ML-random.*

**Fact.** *Let $Z$ be a computable, infinite, co-infinite subset of $\mathbb{N}$. If $A = A_0 \oplus_Z A_1$ is KL-random and $A_1 \in \Delta^0_2$ then $A_0$ is ML-random.*

**Corollary.** *Let $Z$ be a computable, infinite, co-infinite subset of $\mathbb{N}$. If $A = A_0 \oplus_Z A_1$ is KL-random and $\Delta^0_2$ then both $A_0$ and $A_1$ are ML-random. Does this imply that $A$ is ML-random?*