Equivalence of $eF$, $sF$, and $rF$

Thm: $sF$ p-simulates $eF$

proof: let $A_1, ..., A_m$ be an $eF$ proof of $A = A_m$

The proof uses extension rules of the form:

$$y_i \leftarrow \psi_i, \quad i \in [1, k]$$

in the order appearing in the proof, so we have that $y_i$ does not occur in $\psi_j$ for $j \neq i$.

Let $B_i := \psi_i \leftarrow y_i$.

Claim: The $eF$ proof can be modified to give a $sF$ proof of:

$$\prod_{i=1}^k B_i \rightarrow A$$

Use the facts that:
- there are polysize proofs of $\prod B_i \rightarrow B_k$
- the construction of the proof in the deduction that works

$\prod B_i$ is associated left-to-right,

so $\prod B_i \approx ((\cdots (B_1 \wedge B_2) \wedge B_3) \wedge \cdots) \wedge B_k$

So: where does $y_k$ appear in $\prod$? Only in $B_k$, not $B_1 \cdots B_{k-1}$ or $A$.

From $\prod$ obtain a constant-sized proof of:

$$(y_k \leftarrow \psi_k) \rightarrow ((\prod B_i) \rightarrow A)$$

By substitution set: $(\psi_k \leftarrow \psi_k) \rightarrow (\cdots)$
prove: \[ q_k \rightarrow q_R, \text{ use MP to get: } \bigwedge_{l=1}^{k-1} B_i \rightarrow A \]

repeat \( k-1 \) more times, \( \text{QED} \).

Thm: \( eF \) \( p \)-simulates \( sF \)

let \( A_1, \ldots, A_m = A \) be a \( sF \)-proof of \( A \)

We give an \( eF \)-proof of \( A \).

Introduce defns by extention for vars \( x_i^1, \ldots, x_i^p \)

Where \( x_i, \ldots, x_p \) are all vars in \( sF \) proof, \( i = 1 \ldots m \)

Write \( A'_i(x_p) \), replace \( x_j \)'s in \( A_i \) w/ \( x_j^k \)’s

Write \( A'_i(\bar{x}^k) \) to replace \( x_j \)'s in \( A_i \) with \( x_j^k \)'s

Let \( B_k := \bigwedge_{i=1}^{k} \neg A'_i(\bar{x}^k) \) //lines

\( k \in [1, m] \) //lines

is false under \( x_j^k \)'s

\( j \in [1, p] \) //vars

We want to show \( \neg A_m \rightarrow B_k \), for each \( k \in [m] \) of original proof

This will suffice as \( A_i \) is an axiom.

For \( k = m \), take \( x^m_i \rightarrow x_j \) as the extention rule for \( x_j \)

So:

\[
B_k = \bigwedge_{l=1}^{k-1} (\bigwedge_{j=1}^{p} \neg A'_i(\bar{x}^k)) \lor \neg A^k_{i}(\bar{x}^k)
\]
Now suppose we have $x_j^k$'s introduced and $(7,2-2014,3)$ has been proved. We want:
- introduce $x_j^k$'s
- prove $\neg A_m \rightarrow B_k$

Cases:
+ $A_{R+1}$ is an axiom, then take $x_j^k \rightarrow x_j^{k+1}$ as extension rules

$$\begin{align*}
B_k &= \bigvee_{i=1}^{k} (W \rightarrow A_i(\bar{x}^{k+1})) \rightarrow A_k(\bar{x}^{k+1}) \\
\text{Axiom } k
\end{align*}$$

So $B_k \rightarrow W \rightarrow A_i(\bar{x}^{k+1}) \rightarrow \neg A_m \rightarrow W \rightarrow A_i(\bar{x}^{k+1})$

Lem: Let $E(x)$ be a formula w/all occurrences of $x$ shown.
Then $(\psi \rightarrow \psi) \rightarrow (E(\psi) \rightarrow E(\psi))$ has a poly-size F-proof in sizes of $\psi, \psi, E$.

We use this lemma to change variables in the above. So:
- $\neg A_m \rightarrow \bigvee_{i=1}^{k} W \rightarrow A_i(\bar{x}^k)$, i.e. $\neg A_m \rightarrow \neg B_k$
+ $A_{R+1}$ is inferred by MP from $A_i$ and $A_j$
$A_j = A_i \rightarrow A_{R+1}$

Again we can use $x_j^{k+1} \rightarrow x_j^k$

$$(-A_{R+1}) \rightarrow (-A_i \lor \neg A_j)$$ is a tautology w/a short F-proof

This & $B_{R+1}$ implies $B_k$ by a poly-sized F-proof.
ie: $B_{k+1} = \bigvee_{l=1}^{k} A_l \vee \neg (A_{k+1})$

so $B_{k+1} \rightarrow \bigvee_{l=1}^{k} \neg A_l \vee \neg A_i \vee \neg A_j$

So $B_{k+1} \rightarrow \bigvee_{l=1}^{k} \neg A_l$ (successive unwinding & then re-winding to pull out $A_i, A_j$ from the big or)

$+ \neg A_{k+1}$ is inferred by substitution

\[ A_{k+1}(x) \rightarrow \neg A_{k+1}(x^k) \]

Intuition: we have $\neg B_{k+1}$ ie $\bigvee_{l=1}^{k} \neg A_l(x^k)$

ie: $\bigvee_{l=1}^{k}$

Want to set $x_j^{k+1}$ st. one of $A_i, ..., A_k$ is false

$x_j^{k+1} : = \begin{cases} x_j^{k+1} & \text{if } \bigvee_{l=1}^{k} \neg A_l(x^k) \\ \psi_j(x^k) & \text{of } W \end{cases}$

Use extension rule:

$x_j^{k+1} \rightarrow (x_j^{k+1} \land \bigvee_{l=1}^{k} \neg A_l(x^k)) \vee (\psi_j(x^k) \land \bigvee_{l=1}^{k} \neg A_l(x^k))$

Claim: $B_{k+1} \rightarrow B_k$ has a $T$-proof from:

$\bigvee_{l=1}^{k} \neg A_l(x^{k+1}) \rightarrow \bigvee_{l=1}^{k} \neg A_l(x^k)$, from lemma and

$\bigvee_{l=1}^{k} \neg A_l(x^{k+1}) \rightarrow (x_j^{k+1} \rightarrow x_j^{k+1}) \forall j$
and a $F$-proof of

\[ \bigwedge_{l=1}^{k} W \rightarrow A_{l}(\bar{x}^{R_{l}}) \rightarrow A_{\bar{x}}(\bar{x}^{k}) \]

By: (1) $B_{k+1}$ implies $A_{k+1}(\bar{x}^{R_{k+1}})$

\[ \neg A_{i}(\bar{y}_{1}(\bar{x}^{R_{1}}), \ldots, \bar{y}_{p}(\bar{x}^{R_{p}})) \]

(2) $\bigwedge_{l=1}^{k} W \rightarrow A_{l}(\bar{x}^{R_{l}}) \rightarrow (x_{j}^{k} \leftrightarrow \bar{y}_{i}(\bar{x}^{R_{i}}))$

(3) use that lemma

QED

**Thm:** $rF$ p-simulates $sF$

**Proof:** let $A_{1}, \ldots, A_{m} = A$ be a $sF$ proof of $A$

We give 3 proofs & then combine

let $x_{1}, \ldots, x_{p}$ be all vars in proof

(1) $x_{1}, \ldots, x_{p} \rightarrow A$

idea: for each subformula $B$ of $A$,
prove one of:

\[ x_{1}, \ldots, x_{p} \rightarrow B \]

\[ \neg x_{1}, \ldots, x_{p} \rightarrow \neg B \]

Since $A \in \text{TAUT}$ (it has a proof)

we get $x_{1}, \ldots, x_{p} \rightarrow A$ (actually using that what we've proved in $rF$ is true)
(2) \( \forall x_1, \ldots, x_p \rightarrow A \)

(3) let Distinct \((x_1, \ldots, x_p)\) be
\[ \neg x_1 \land \ldots \land x_p \land \neg x_p \]

prove Distinct \((x_1, \ldots, x_p) \rightarrow A \)

There is a polyline \(T\)-proof of
\[ (x_1, \ldots, x_p) \lor (\neg x_1, \ldots, x_p) \lor \text{Distinct}(x) \]

Successively prove Distinct \((x) \rightarrow A_i, i = 1, \ldots, m \)

If \(A_i\) is an axiom: no problem \(D := \text{Distinct}(x)\)

If \(- A_i\) is from \(MP, A_j, A_k\)
\[ (D \rightarrow A_j) \rightarrow (D \rightarrow A_k) \rightarrow (D \rightarrow A_i) \] is an axiom

get \(D \rightarrow A_i\) by \(MP\)

If \(A_i\) is from substitution:
\[ A_j(x_2) \]

appears in original proof
\[ A_i(\varphi) \] where \(A_i = A_j(\varphi)\)

idea: can infer \(A_j(x_2) \lor s = 1, \ldots, p\), under hypothesis that \(D\) holds, at least \(1 x_s\) is true
\(\exists x\) a different \(x_s\) is false.

Thus: \(\neg A_j(x_2) \land \neg A_j(x_3) \land \neg A_j(x_3) \land \ldots \land \neg A_j(x_3) \)

if \(\varphi\) is true, \(A_j(\varphi)\) follows

\[ \neg A_j(x_3) \] from \(A_j(x_2)\)

if \(\varphi\) is false, \(A_j(\varphi)\) follows
from \(A_j(x_3)\)
Concern: \( D(x_1, \ldots, x_p) \rightarrow A_i(x_e) \)

rename

\( D(x_1, \ldots, x_{k-1}, x_5, \theta) \rightarrow A_i(x_5) \)

but if \( D \) of everyone holds might've dropped our distinct var to get \( A_i(x_5) \)

\( \equiv \) Different?

Assume \( \neg \varphi \)

\( \Rightarrow \) Assume \( x_e, \neg x_5 \) (ie \( x_5 \) is false)

if \( D(x_1, \ldots, x_{k-1}, x_5', x_5', \ldots, x_p) \) then

\( A_i(x_5') \) so \( A_i(\varphi) \)

if \( \neg D(x_1, \ldots, x_{k-1}, x_5', x_5', \ldots, x_p) \)

then go back to fact that \( A_m \) is valid

proof: infer \( D(\ldots, \neg x_e \ldots) \rightarrow A_i(x_5') \) \( \forall x_5' \)

give T-proof of those \& \( D \rightarrow A_i(x_e) \)

of (1) \( x_e \land \neg \varphi \land \neg x_5 \rightarrow (D \rightarrow A_i(\varphi)) \)

- if \( D(x_1, \ldots, x_5', \ldots, x_p) \)

then \( A_i(x_5) \) \& by lemma \( A_i(\varphi) \)

- if \( \neg D(x_1, \ldots, x_5', \ldots, x_p) \)

then \( A_i(\varphi) \) from \( \neg D \rightarrow B \) for each subformula \( \neg B \) \& \( B \) of \( A_i \)
OPEN: permutation $F$: can it p-sim $sF$?