Higher polynomial local search for fragments of bounded arithmetic

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(joint work with Arnold Beckmann)
Bounded Arithmetic and Provably Total Functions.

Recall some systems:

- $PV$ - Induction on polynomial time predicates (Cook 1975)
- $I\Delta_0$ - Induction on linear time hierarchy predicates (Parikh, 1971).
- $\Omega_1$ - Totality of $x^{\log x}$ ("smash" function, #)
- $T^b_k$ - Induction on $\Sigma^b_k$-predicates, at $k$-th level of polynomial time hierarchy.
- $T^1_2$ - Induction on NP predicates. (Buss 1985)
- $S^b_2$ - Length or polynomial induction on $\Sigma^b_k$-predicates. [ibid]

$PV \preceq S^1_2 \subseteq T^1_2 \preceq S^2_2 \subseteq T^2_2 \preceq S^3_2 \subseteq \cdots \cdots \cdots I\Delta_0 + \Omega_1.$

$S^{k+1}_2$ is $\forall \Sigma^b_{k+1}$-conservative over $T^b_k$.

Analogy (weak): $S^b_2 \approx I\Sigma_k$ and (polynomial time)$\approx$(primitive recursive).
### Provably total functions.

<table>
<thead>
<tr>
<th>Theory</th>
<th>Graph</th>
<th>(Multi)Function class</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^1_2$</td>
<td>$\Sigma^b_1$-defined</td>
<td>$P$, polynomial time functions</td>
</tr>
<tr>
<td>$T^1_2$</td>
<td>$\Sigma^b_1$-defined</td>
<td>PLS, polynomial local search multifunctions.</td>
</tr>
<tr>
<td>$S^k_2$</td>
<td>$\Sigma^b_k$-defined</td>
<td>$P^{\Sigma^b_{k-1}}$ functions.</td>
</tr>
<tr>
<td>$T^k_2$</td>
<td>$\Sigma^b_k$-defined</td>
<td>$PLS^{\Sigma^b_{k-1}}$ multifunctions.</td>
</tr>
<tr>
<td>$S^{k+1}_2$</td>
<td>$\Sigma^b_k$-defined</td>
<td>$PLS^{\Sigma^b_{k-1}}$ multifunctions.</td>
</tr>
<tr>
<td>$T^2_2$</td>
<td>$\Sigma^b_1$-defined</td>
<td>Colored PLS. (Krajíček-Skelley-Thapen, 2006)</td>
</tr>
<tr>
<td>$T^k_2$</td>
<td>$\Sigma^b_1$-defined</td>
<td>Herbrand analysis (Pudlák, 2006).</td>
</tr>
<tr>
<td></td>
<td>”</td>
<td>$k$-turn games (Skelley-Thapen, 200?).</td>
</tr>
<tr>
<td>$T^2_2$</td>
<td>$\Sigma^b_1$-defined</td>
<td>Iterated PLS (Arai, 200?)</td>
</tr>
<tr>
<td>$T^k_2$</td>
<td>$\Sigma^b_i$-defined</td>
<td>$\Pi^b_k$-PLS with $\Pi^b_{i-1}$-goal ($1 \leq i \leq k$) - (this talk)</td>
</tr>
</tbody>
</table>

Polynomial Local Search (PLS) problems.

(Johnson-Papadimitriou-Yannakakis, 1988). A PLS problem defines a total multifunction $f$ with polynomial time graph $f(x) = y$. It has:

- A set $F(x) := \{ s : F(x, s) \}$ of feasible points $\leq t(x)$,
- An initial point $i(x) \in F(x)$.
- A cost function $c(x, s)$.
- A neighborhood function $N(x, s)$.
- $F$, $N$, $c$, $i$ and $t$ are polynomial time.
- For all $s \in F(x)$, $N(x, s) \in F(x)$ and
  
  either $N(x, s) = s$ or $c(N(x, s)) < c(s)$.
- If $s \in F(x)$ and $N(x, s) = s$, then $y = (s)_1$ is a value of $f(x)$.

$f(x) = y$ holds if and only if $s \in F(x)$ and $N(x, s) = s$ and $(s)_1 = y$.

Algorithm: Start with $s = i(x)$ and iterate $N$. Is in PSPACE.
Open question: Are PLS problems in P?
\( \Pi^p_k \)-PLS — relativizing PLS

\( (PLS^{\Pi^p_k} \) has \( F, c, N, i \) in \( P^{\Pi^p_k} \).)

\( \Pi^p_k \)-PLS has \( F \in \Pi^p_k \), but \( N, c, i \) are polynomial time.

\( \Pi^p_k \)-PLS problems by definition satisfy (\( \alpha \))-\( (\delta) \):

(\( \alpha \)) \( \forall x \forall s (F(x, s) \rightarrow |s| \leq d(|x|)), \ d \) a polynomial.

(\( \beta \)) \( \forall x (F(x, i(x))). \)

(\( \gamma \)) \( \forall x \forall s (F(x, s) \rightarrow F(x, N(x, s))). \)

(\( \delta \)) \( \forall x \forall s (N(x, s) = s \lor c(x, N(x, s)) < c(x, s)). \)

Defines a multifunction \( f(x) = y \) by:

\[
    f(x) = y \iff (\exists s \leq 2^{d(|x|)}) [F(x, s) \land N(x, s) = s \land y = (s)_1].
\]

Same algorithm applies, and is still in PSPACE.
$\Pi^p_k$-PLS with $\Pi^p_g$-goal $G$

A $\Pi^p_k$-PLS problem with $\Pi^p_g$-goal $G(x, s)$ satisfies the additional property:

$$(\epsilon) \ \forall x \forall s (G(x, s) \leftrightarrow [F(x, s) \land N(x, s) = s]).$$

The graph of the multifunction can now be defined by

$$f(x) = y \iff (\exists s \leq 2^{d(|x|)}) [G(x, s) \land y = (s)_1],$$

so $f$ has a $\Sigma^b_{g+1}$-definition.

**Formalized $\Pi^p_k$-PLS problems**: The predicates $F$ and $G$ are given by $\Pi^b_k$- and $\Pi^b_g$-formulas, $N, i, c$ are polynomial time functions, and the base theory $S^1_2$ proves conditions $(\alpha)$-$(\epsilon)$.

Formalized $\Pi^p_k$-PLS problems are called $\Pi^b_k$-PLS problems.
Existence of solutions to $\Pi_k^b$-problems

**Thm 1.** Let $\mathcal{P}$ be a $\Pi_k^b$-PLS problem. Then $T_2^{k+1}$ proves that, for all $x$, $\mathcal{P}(x)$ has a solution:

$$\forall x \exists s (F(x, s) \land N(x, s) = s),$$

or

$$\forall x \exists s (G(x, s)).$$

This is a $\Sigma_{k+1}^b$ (resp., $\Sigma_{g+1}^b$) definition of a multifunction.

**Pf.** Use $\Sigma_{k+1}^b$-minimization to find the least $c_0$ satisfying

$$\exists s \leq 2^{d(|x|)} (c_0 = c(x, s) \land F(x, s)).$$
Exact characterization of $\Sigma^b_i$-definable functions of $T_{2}^{k+1}$.

Main Thm 2 Let $0 \leq g \leq k$ and $A(x, y) \in \Sigma^b_{g+1}$. Suppose

$$T_{2}^{k+1} \vdash (\forall x)(\exists y)A(x, y).$$

Then there is a $\Pi^b_k$-PLS problem $\mathcal{P}$ with $\Pi^b_g$-goal $G$ such that $S_{2}^{1}$ proves

$$\forall x\forall s (G(x, s) \rightarrow A(x, (s)_1)).$$

Note that the conclusion is provable in $S_{2}^{1}$, but $T_{2}^{k+1}$ is needed to prove the existence of $s$.

For $k = g = 0$, states that the $\Sigma^b_1$-definable functions of $T_{2}^{1}$ are in PLS.
Why formalization in $S^1_2$ is important (#1)

Consider a total multifunction defined by $(\forall x)(\exists y \leq t)A(x, y)$, where $A \in \Delta^b_0$. Here is a $\Pi^p_1$-PLS search problem for it:

- Initial function: $i(x) = 0$.
- Cost function: $c(x, y) = t - y$.
- Neighborhood function: $N(x, y) = \begin{cases} y & \text{if } A(x, y) \text{ or } y \geq t(x) \\ y + 1 & \text{otherwise.} \end{cases}$
- Feasible set: $F(x, y) \Leftrightarrow [A(x, y) \lor (\forall y' < y)(\neg A(x, y'))] \land y \leq t(x)$.
- Goal: $G(x, y) \Leftrightarrow A(x, y) \land y \leq t(x)$.

This is a correct $\Pi^p_1$-PLS problem independently of provability in $T^k_{2+1}$. But it is not formalizable in $S^1_2$, so is not a $\Pi^b_1$-PLS problem.
**Thm 2’s proof strategy:** Fix $k \geq 0$.

**Defn** Let $A(\vec{c}) \in \Sigma^b_{k+1}$. $Wit_A(u, \vec{c})$ is a $\Pi^b_k$-formula that states $u$ codes values for the outermost existential quantifiers of $A(\vec{c})$ making $A(\vec{c})$ true.

**Witnessing Lemma.** If $T_{2}^{k+1}$ proves a sequent

$$\Gamma \rightarrow \Delta$$

of $\Sigma^b_{k+1}$-formulas with free variables $\vec{c}$, then there is a multifunction $f$ defined by a $\Pi^b_k$-PLS problem such that

$$S_2^1 \vdash Wit_{\land \Gamma}(u, \vec{c}) \land y = f(\langle u, \vec{c} \rangle) \rightarrow Wit_{\lor \Delta}(y, \vec{c}).$$

Proof is by induction on length of a free-cut free proof. Part of the proof requires finding a $\Pi^b_k$-PLS problem for determining the truth of a $\Pi^b_k$-formula.
Tools for proving the Witnessing Lemma include the following.

**Π^b_k-PLS problems:**

- are closed under polynomial time operations
- closed under composition
- are closed under “pseudoiteration”: exponentially long, polynomial space bounded iteration that preserves a **Π^b_k**-property. (Needed to handle induction inferences.)
- can decide the truth of **Π^b_k**-properties. (Needed to handle ∀ ≤:right inferences.)

For deciding truth of **Π^b_k**-formulas, see next page ...
Determining truth of a $\Pi^b_k$-formula

**Lemma** Let $A(x) = (\exists y \leq t) B(y, x) \in \Sigma^b_k$, with $B \in \Pi^b_{k-1}$. There is a $\Pi^b_k$-PLS problem $\mathcal{P}_A$ that explicitly determines the truth of $A(x)$ by computing

$$\mathcal{P}_A(x) = \begin{cases} 
    \langle 0, t + 1 \rangle & \text{if } \neg A(x) \\
    \langle 1, i \rangle & \text{if } i \leq t \text{ is the least value s.t. } B(i, x).
\end{cases}$$

**Pf.** Define initial function $i(x) := \langle 0, 0 \rangle$. Define

$$N(x, \langle 0, i \rangle) = \begin{cases} 
    \langle 0, i + 1 \rangle & \text{if } \neg B(i, x), \ i \leq t. \\
    \langle 1, i \rangle & \text{otherwise}
\end{cases}$$

$N(x, s) = s$ for all other $s$.

For $k > 1$, determining $\neg B(i, x)$ involves calling $\mathcal{P}_B$, a $\Pi^b_{k-1}$-PLS problem. Then define $F(x, \langle 0, i \rangle) \iff i \leq t + 1 \land (\forall j < i)(\neg B(j, x))$ and $F(x, \langle 1, i \rangle) \iff i \leq t \land B(i, x) \land (\forall j < i)(\neg B(j, x))$.

$F(x, s)$ is false for all other $s$. Note $F \in \Pi^b_k$.

Cost function $c(x, \langle j, i \rangle) = t + 1 - i$.  

11
Skolemization: A stronger version of $\Pi^b_k$-PLS witnessing

**Skolemization:** For a Boolean combination of formulas, create equivalent prenex form by the following procedure. Find all outermost blocks of quantifiers not yet processed. Bring out all universal ones first, then all existential ones. Repeat until in prenex form. Then Skolemize with terms.

**Example:** If $F$ is $\forall y \exists z F_0(y, z)$, then $(\gamma)$ is Skolemized as follows:

Recall $(\gamma)$ is: $\forall x, s (F(x, s) \rightarrow F(x, N(x, s)))$.

Prenex form: $\forall x, s, y_2 \exists y_1 \forall z_1 \exists z_2 (F_0(x, s, y_1, z_1) \rightarrow F_0(x, s, y_2, z_2))$.

Skolem form: $\forall x, s, y_2, z_1 (F_0(x, s, r(x, s, y_2), z_2) \rightarrow F_0(x, s, y_1, t(x, s, y_2, z_1)))$.

where $r$ and $t$ are terms (over the language $0, S, +, \cdot, \div, MSP$ that allows simple fixed-length sequence coding.) $r$ and $t$ are polynomial time.
Defn A $\Pi^b_k$-PLS problem with $\Pi^b_g$ goal is *formalized in Skolem form* provided the functions $N$, $c$, and $i$ are defined by terms, the formulas $F$ and $G$ are “strict” formulas (with no sharply bounded quantifiers in front of bounded quantifiers, etc.) and provided $S^1_2$ proves all the conditions $(\alpha)$-$(\delta)$ plus

$$(\epsilon') \forall x \forall s (G(x, s) \rightarrow [F(x, s) \land N(x, s) = s])$$

$$(\epsilon'') \forall x \forall s ([F(x, s) \land N(x, s) = s] \rightarrow G(x, s))$$

in Skolem form using terms as Skolem functions.

Thm If $\mathcal{P}$ is formalized in Skolem form, it is also formalized in the usual form.

Pf. This is trivial.
Exact characterization revisited, Skolemized form

**Main Thm 3.** Let $0 \leq g \leq k$ and $A(x, y) \in \Sigma^b_{g+1}$. Suppose

$$T_{2}^{k+1} \vdash (\forall x)(\exists y)A(x, y).$$

Then there is a $\Pi^b_k$-PLS problem $\mathcal{P}$ with $\Pi^b_g$-goal $G$ which is formalized in Skolem form such that $S_{1}^{1}$ proves a Skolemization of:

$$\forall x \forall s(G(x, s) \rightarrow A(x, (s)_1)).$$
The proof of the theorem is similar to before, but much more delicate.

**One potential problem.** For $A \in \Pi^b_k$, $k \geq 2$, the formula

$$A \rightarrow A \land A$$

may not be provable in Skolem form by $S^1_{2}$.  

This is needed to handle (implicit) contractions in the free-cut free $T^i_{2}$-proof.

**Solution:** Use $\mathcal{P}_A$, the $\Pi^b_k$-PLS problem that determines the truth of $A$. The formula

$$A(x) \land y = \mathcal{P}_A(x) \rightarrow A(x) \land A(x)$$

is provable in Skolem form by $S^1_{2}$. 


A separation conjecture

We can set up a “generic” Skolemized $\Pi^b_k$-PLS problem with $\Pi^b_0$-goal as follows: Adjoin a new predicate symbol for $G$ and a new predicate symbol $F_0$ for the sharply bounded subformula of $F$, and adjoin new functions symbols which are used as Skolem functions for the $\Pi^b_k$-PLS problem’s defining conditions.

Then, the Skolemized definition of the $\Pi^b_k$-PLS problem can be expressed as a single $\forall \Delta^b_0$-formula.

Encoding the new functions and predicates by a single new predicate $\alpha$, we can encode this $\forall \Delta^b_0$-formula as a single $\forall \Delta^b_0$-formula $\forall x \Psi(x, \alpha)$. 
Consider the formula

\[ \forall x \Psi(x) \rightarrow \forall x \exists y \leq x (y = N(x, y) \land G(x, y)) \].

By the relativized version of the main theorems, it is provable in \( T_{2}^{k+1}(\alpha) \).

On the other hand, by the conjectured properness of the bounded arithmetic and polynomial time hierarchies, we expect this is not provable in \( T_{2}^{k}(\alpha) \).

This gives a single \( \forall \Sigma_{1}^{b}(\alpha) \)-formula that is known to be provable in \( T_{2}^{k+1}(\alpha) \) but conjectured to not be provable by \( T_{2}^{k}(\alpha) \).

**Why formalization in \( S_{2}^{1} \) is important (#2).** Since the Skolem functions are polynomial time (in fact, are given by simple terms), they can be conservatively added to \( S_{2}^{1}, T_{2}^{k}, \) etc., and can be used freely in induction axioms. Thus, it is reasonable to allow \( T_{2}^{k+1}(\alpha) \) use the new predicate \( \alpha \) freely in induction axioms.
Conjectured separation for constant depth propositional proofs

By using the Paris-Wilkie translation, we get a conjectured separation for bounded depth propositional proof systems. The Paris-Wilkie translation converts existential and universal quantifiers to OR’s and AND’s, and atomic formulas to either True or False or, for $\alpha(t)$, to a propositional variable $p_t$.

A depth $k$ Tait system has sequents of formulas of depth $k$, where depth is measured by alternations of AND’s and OR’s. (Poly)logarithmic depth fanin at the bottom level counts as a 1/2 depth. The $T_{2^{k+1}}^k(\alpha)$ proof of Thm 1 translates to a depth $k − 1$ proof by the Paris-Wilkie translation (after several careful transformations).

The end result gives, for each $x \in \mathbb{N}$, a set $\Xi_k$ of sequents of literals such that $\Xi_k$ is known to have depth $k − 1$ Tait-style refutations, but is conjectured to not have depth $k − 1\frac{1}{2}$ depth refutations.
Some open problems

1. Can depth $k$ propositional proofs be separated from depth $k - 1$ proofs, for low depth tautologies?

2. Is there a non-uniform version of the witnessing theorems for $T^k_2$ that will apply to depth $k - \frac{1}{2}$ propositional proofs?

3. Are there good analogues of Thms 2 or 3 for fragments of Peano arithmetic?

Advertisements:
- Problem session (Beckmann, Buss, and ? — today at 2:00.)
- Friday at 9:00am, Arnold will speak on related material?
Happy Birthday, Stan!