

Problems proposed by Steve Butler

Mathematics Magazine **76** (April 2003), 151.

1668. Let f be a real valued function defined on an open interval containing $[a, b]$. Assume that f has a continuous second derivative on I and there is a single line tangent to the graph of $y = f(x)$ at $(a, f(a))$ and $(b, f(b))$. Prove that if $f''(x)$ is not identically zero on (a, b) , then $f''(x)$ must change sign at least twice on (a, b) .

Solution published in *Mathematics Magazine* **77** (April 2004), 159.

Alternative solution:

There is a single line tangent at $(a, f(a))$ and $(b, f(b))$ if and only if $f'(a) = f'(b)$ (same slope) and $f(a) - af'(a) = f(b) - bf'(b)$ (same y -intercept). Since

$$\begin{aligned}\int_a^b (Cx + D)f''(x) dx &= (C(f(x) - xf'(x)) + Df'(x)) \Big|_a^b \\ &= C((f(b) - bf'(b)) - (f(a) - af'(a))) + D(f'(b) - f'(a)),\end{aligned}$$

it follows that there is a single tangent line at $(a, f(a))$ and $(b, f(b))$ if and only if $\int_a^b (Cx + D)f''(x) dx = 0$ for all constants C and D .

If f'' never changed sign on (a, b) then since $f'' \not\equiv 0$ we have $\int_a^b f''(x) dx \neq 0$ and so we could not have the tangent line. If f'' changed sign only once then for some $a < c < b$ we would have $(x - c)f''(x)$ either nonnegative or nonpositive on (a, b) , in either case we would again have that $\int_a^b (x - c)f''(x) dx \neq 0$ and so again we could not have the tangent line. So it must be that f'' changes sign at least twice.

The American Mathematical Monthly **110** (Aug.-Sep. 2003), 637.

11030. Show that for $d < -1$ there are exactly two real-valued functions f such that for all real x and y $f(x + y) - f(x)f(y) = d \sin x \sin y$.

Solution published in *The American Mathematical Monthly* **112** (April 2005), 371.

More information at <http://www.math.ucsd.edu/~sbutler/PDF/cauchy.pdf>

Mathematics Magazine **78** (October 2005), 325.

1730. Let A and B be symmetric, positive semi-definite matrices such that $A + B$ is positive definite, and let $\|\mathbf{y}\|$ denote the usual 2-norm of the vector \mathbf{y} . Prove that for all $\mathbf{x} \neq \mathbf{0}$,

$$\|(I - A)(I + A)^{-1}(I - B)(I + B)^{-1}\mathbf{x}\| < \|\mathbf{x}\|.$$

Solution published in *Mathematics Magazine* **79** (October 2006), 313-314.

More information at http://www.math.ucsd.edu/~sbutler/PDF/pos_norm.pdf

The American Mathematical Monthly **114** (January 2007), 77.

11265. (*Proposed jointly with Jia Mao.*) For a graph G , let $c(G)$ denote the minimum k such that every edge lies in a cycle of length at most k ($c(G)$ is infinite when G has a cut-edge). Prove that for $n \geq 3$ the minimum of $|E(G)| + c(G)$, taken over all n -vertex connected graphs, is $n + \lceil 2\sqrt{n-1} \rceil$.

Solution to appear in *The American Mathematical Monthly*.

Mathematics Magazine **80** (February 2007), 77.

1761. For integer $n \geq 2$ define the sets

$$A(n) = \{(k, l) : 1 \leq k \leq l \leq n, k + l \leq n, \text{ and } \gcd(k, l) = 1\}$$

$$B(n) = \{(k, l) : 1 \leq k \leq l \leq n, k + l > n, \text{ and } \gcd(k, l) = 1\},$$

where $\gcd(k, l)$ denotes the greatest common divisor of the integers k and l . Prove that $A(n)$ and $B(n)$ have the same cardinality.

Two different solutions published in *Mathematics Magazine* **81** (February 2008), 64-66.