TANGENT LINE TRANSFORMATIONS: OR THERE AND BACK AGAIN

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Introduction

In the first semester of Calculus students learn to take a curve and find all of the curves tangent lines. Now consider the converse problem, if given all of the curves tangent lines can we determine the original curve that they came from?

This question has been addressed before with some success (see [2]). In this note we will present a new method that is both beautiful in its simplicity and surprising in its method.

In this note we will first develop a way to represent all of the tangent lines to the curve simultaneously and then see how this process of finding this representation actually gives a way to reconstruct the curve.

Dealing with lots of tangent lines

The first problem with reconstructing the curve from the tangent lines is how to deal with the tangent lines. When we are dealing with just one or two tangent lines then we can easily just represent our tangent lines by lines. But now imagine that we were to draw every single tangent line in as a line on our graph. Soon our paper would be a black mass of lines and we would lose all sense of where our tangent lines were.

So let us consider a different way of representing these lines. Lines can almost always be described by two values, for example the slope and the y-intercept. So instead of drawing a line for each tangent line let us associate each line with a point determined by two particular values. That will clean up our picture immensely. So we have the following definitions.

Definition. A *line transformation* is a rule that associates every line with either a point or it will say that the line can not be associated with any point. A *tangent line transformation* is a line transformation applied to all of the tangent lines of a curve.

There are many different line transformations that can be used (and hence many different tangent line transformations). Examples of line transformations include associating a line with the slope and the y intercept or associating a line with the x-intercept and the reciprocal of the slope or associating a line with both intercepts and so on.

Here there is some flexibility as to what transformation we use (keep in mind of course that different transformations will produce different results). For our purpose we will use the transformation that will associate the slope of the tangent line with the x coordinate of the point and the y-intercept of the line with the y coordinate of the point. If we have any line where the slope is undefined (i.e. vertical lines) we will not associate that with any point.

Parameterizing our transformation

With our tangent line transformation in place we can explore how the transformation will act on a parameterized curve, $(x(t), y(t))$ in the plane. When both $x'(t)$ and $y'(t)$ exist and at least one is non-zero then the tangent line to our parameterized curve is given by the following (this can be found in nearly every calculus book, such as [1]),

$$x'(t)[y - y(t)] - y'(t)[x - x(t)] = 0.$$

When $x'(t) \neq 0$ this can be rearranged into slope-intercept form,

$$y = \left[\frac{y'(t)}{x'(t)}\right] x + \left[y(t) - x(t)\frac{y'(t)}{x'(t)}\right].$$

From this last relationship we can solve for the slope and the y-intercept of the tangent line.

$$\text{slope} = \frac{y'(t)}{x'(t)} \qquad y\text{-intercept} = y(t) - x(t)\frac{y'(t)}{x'(t)}$$

In particular, we can now parameterize the values of the slope and the y-intercept (with discontinuities where the tangent lines do not exist or are vertical) from our parameterization of the original curve. In other words, the tangent line transformation of a parameterized curve is also a parameterized curve.

To distinguish the parameterization of the result from the original we will adopt the * notation. So we have that the line tangent to the curve at the point $(x(t), y(t))$ will be transformed to the point $(x^*(t), y^*(t))$ in the following way,

$$(x^*(t), y^*(t)) = \left(\frac{y'(t)}{x'(t)}, y(t) - x(t)\frac{y'(t)}{x'(t)}\right)$$

Note if the functions $x(t)$ and $y(t)$ have continuous derivatives (i.e. they are C^1), then the parameterization of the result of the transformation will be continuous everywhere that $x'(t) \neq 0$ and in particular if $x'(t) \neq 0$ for all t then the parameterization of the result of the transformation is everywhere continuous.

A visual example

With an idea of what a tangent line transformation let us see the transformation in action. In Figure 1 there is a curve (on the left) and the results of the tangent line transformation to that curve (on the right).

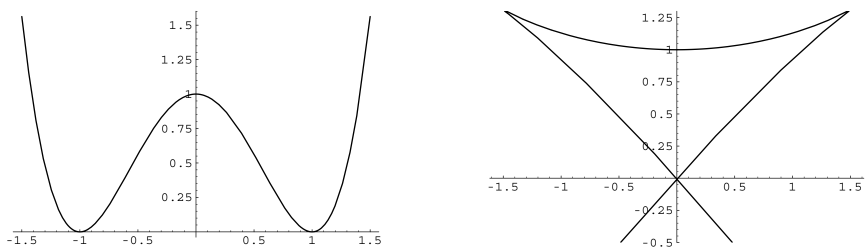

Figure 1. A function and its tangent line transformation

The original curve was parameterized by,

$$(x(t), y(t)) = \left(t, t^4 - 2t^2 + 1\right) \qquad \text{where } t \in (-\infty, \infty).$$

By applying the methods outlined previously we can find the parameterization of the tangent line transformation and get the following,

$$(x^*(t), y^*(t)) = \left(4t^3 - 4t, -3t^4 + 2t^2 + 1\right) \qquad \text{where } t \in (-\infty, \infty).$$

The result of the tangent line transformation actually contains quite a bit of information about the original curve. For example the places where the result of the transformation crosses itself represents tangent lines which are tangent at more than one point. Later, we shall see that the cusps in the transformation correspond to the inflection points on the original curve.

But now returning to our original goal, that of reconstructing a curve from the set of its tangent lines, we have to wonder if there is *enough* information contained in the results of the transformation to find the original curve, so we now turn our attention to answering this question.

Once, twice...

We noted earlier that if our function was C^1 that the result of the transformation was continuous everywhere that $x'(t) \neq 0$. If we now consider functions that are C^2 then the result of the transformation is not only continuous but also differentiable and so we can apply the tangent line transformation to the results of applying the tangent line transformation. So let us apply this transformation twice and see what, if anything occurs.

First we will work through a simple example, a parabola in the plane which we can parameterize by $(x(t), y(t)) = (t, at^2 + bt + c)$. If we apply the transformation the first time we will get the following (again we omit the calculations, though you are more then welcome to verify these at your leisure),

$$(x^*(t), y^*(t)) = (2at + b, -at^2 + c).$$

Now let us take the resulting curve, and in particular the resulting parameterization that we have for the curve, and apply the transformation again. To denote applying the transformation twice we will use a ** notation. So starting with $(x^*(t), y^*(t)) = (2at + b, -at^2 + c)$ we will get the following,

$$(x^{**}(t), y^{**}(t)) = (-t, at^2 + bt + c).$$

This is *almost* the original curve, the only difference being that it has been flipped around the y-axis.

That the results of applying the transformation twice gives back the original curve flipped around the y-axis is a general result which we shall now prove.

The general theorem

Theorem. *Given a curve which is parameterized by $x(t)$, $y(t)$ with t going over some range, then on any open interval of the domain upon which both x and $y \in C^2$, $x'(t) \neq 0$ and $x'(t)y''(t) - y'(t)x''(t) \neq 0$ the following holds:*

$$(x^{**}(t), y^{**}(t)) = (-x(t), y(t))$$

Proof: Since $x'(t) \neq 0$ in the open interval and x, $y \in C^2$ then for every value of t in the open interval both x^* and y^* exist. Further, since x^* and y^* are composed of differentiable functions then we know that they themselves are differentiable. Finally, note that

$$(x^*(t))' = \left(\frac{y'(t)}{x'(t)}\right)' = \frac{x'(t)y''(t) - y'(t)x''(t)}{(x'(t))^2} \neq 0$$

Therefore, $x^*{*}$ and $y^*{*}$ will exist at every point in the interval.

So applying the transformation twice we will get the following:

$$\begin{aligned}
x^{**}(t) &= \frac{(y^*)'(t)}{(x^*)'(t)} = \frac{\left(y(t) - x(t)\frac{y'(t)}{x'(t)}\right)'}{\left(\frac{y'(t)}{x'(t)}\right)'} \\
&= \frac{y'(t) - x'(t)\frac{y'(t)}{x'(t)} - x(t)\left(\frac{y'(t)}{x'(t)}\right)'}{\left(\frac{y'(t)}{x'(t)}\right)'} = -x(t) \\
y^{**}(t) &= y^*(t) - x^*(t)\frac{(y^*)'(t)}{(x^*)'(t)} \\
&= \left[y(t) - x(t)\frac{y'(t)}{x'(t)}\right] - \left[\frac{y'(t)}{x'(t)}\right][-x(t)] \\
&= y(t) - x(t)\frac{y'(t)}{x'(t)} + x(t)\frac{y'(t)}{x'(t)} = y(t).
\end{aligned}$$

Combining these results conclude the proof.□

This theorem shows that the square of this transformation is a reflection around the y-axis. This is what will allow us to reconstruct a curve from the tangent lines.

Let us quickly see what this theorem has to say about the special case when we have $y = f(x)$ where $f(x) \in C^2$. This curve can be parameterized by $(x(t), y(t)) = (t, f(t))$. Since $x'(t) = 1$ and $x''(t) = 0$ the conditions of the theorem will be satisfied on any open interval upon which $f''(t) \neq 0$. So in particular we can apply the transformation twice except when the second derivative is zero.

Recall the cusps that were in Figure 1, at these points on the curve resulting from the transformation we cannot take the derivative, and so they correspond to points on the original curve where we cannot apply the transformation twice. This is why the cusps in the resulting curve from the transformation correspond to places where the second derivative of the original curve was zero, which include the inflection points.

Reconstructing a curve from its set of tangent lines

With the theorem in hand let us apply it to see how we can reconstruct a curve from its tangent lines. Of course the theorem does not apply to every function and so the method will not work on every curve. So for the following procedure we will assume that we have a curve which is continuous, closed and satisfies the conditions of the theorem on a dense set. Then on such a curve the process of reconstruction is as follows.

First, since we have all of the tangent lines to the curve we can apply the tangent line transformation to this curve (keep in mind that the tangent line transformation does not act on the curve directly but rather on the tangent lines of the curve, which in this case is what we are given). This will produce a set of points in the plane (note that the results of a tangent line transformation need not be connected and could be composed of infinitely many disconnected curves and hence the reason to say "set").

Second, since the conditions of the theorem are satisfied on a dense set of our domain, then we can find many of the tangent lines to the resulting curve(s) and so we find all the tangent lines to our new set and again apply the tangent line transformation. This will recover points of our original curve (reflected around the y-axis) on that dense set of our domain for which the conditions are satisfied.

Third, it is possible that the previous step will miss some points on our original curve, and so to recover any points that we missed we take the closure of the result from the second step. Since our original curve was continuous and we have already recovered points onf the curve on a dense set of the domain we will pick up any missing points. Further since our original curve was closed, we will not pick up any new points. This will produce a curve that is a reflected copy of the original curve.

Finally, reflecting the results across the y-axis will give us back our original curve. This finishes the reconstruction.

Some questions left to be answered

We have now laid out the basic ideas of tangent line transformations. A definite plus for this transformation is the strange and beautiful pictures that it can produce as a step in the process (such as in Figure 1). To get a feel for how this transformation works it is best to test drive the transformation on some curves and explore the results.

This has been a short introduction to tangent line transformations and there are some questions that could be more fully explored. These include.

- What is happening geometrically when we apply the transformation twice? In particular, why does it flip around the y-axis?
- The transformation that we have explored flips around the y-axis, if we had used as a line transformation a rule that associates the x-intercept with the x coordinate and the reciprocal of the slope with the y coordinate it would flip it around the x-axis. Are there any others with similar behavior?
- What information can we get about the original graph by studying the results of the tangent line transformation? What if we had used a different tangent line transformation?
- Is there a three dimensional equivalent? What about higher dimensions?

References

[1] D. Berkey and P. Blanchard, Calculus (3rd ed), Saunders, 1992.
[2] A. Horwitz, "Reconstructing a function from its set of tangent lines," *Amer. Math. Monthly* vol. 96, no. 9, pages 807-813, 1989.

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