The number of ordered tuples with no global factor

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1 Introduction

In this note we consider the problem of enumerating the number of ordered k-tuples with entries in $[1,n]$ and with no global factor other than 1. Some enumeration for small values of n for $k=2,3,4,5$ has been done previously (see `A002088`, `A015631`, `A015634` and `A015650` in [3]). We show that that the number of these k-tuples are the partial sums of the Moebius transform of the binomial coefficients. So for example, the number of 3-tuples for fixed n (`A015631`) is the partial sum of the first n terms in the Moebius transform of the triangular numbers (`A007438`). (A previously unknown connection.)

We also give an explicit formula using binomial coefficients and Mertens function (which has been closely studied due to its ties with the Riemann hypothesis (see [2])) to enumerate these objects for all k and n.

2 Main result

Theorem 1. *Let* $\gcd(a_1,a_2,\ldots,a_k)$ *denote the greatest global divisor of* $a_1,a_2,\ldots,a_k$*, let* $\mu(n)$ *denote the Moebius function and* $M(n)=\sum_{i=1}^{n}\mu(i)$ *denote Mertens function. Then for* $n\geq 1$

$$\begin{aligned}&\left|\{(a_1,a_2,\ldots,a_k): 1\leq a_1\leq a_2\leq\cdots\leq a_k\leq n,\ \gcd(a_1,a_2,\ldots,a_k)=1\}\right| =\\ &\quad\sum_{m=1}^{n}\sum_{d|m}\mu(m/d)\binom{d+k-2}{k-1}=\sum_{i=1}^{n}\mu(i)\binom{\lfloor n/i\rfloor+k-1}{k}=\sum_{i=1}^{n}M(\lfloor n/i\rfloor)\binom{i+k-2}{k-1}.\end{aligned}$$

In the first expression on the second line the terms $\sum_{d|m}\mu(m/d)\binom{d+k-2}{k-1}$ correspond to the Moebius transform of the binomial coefficients, showing that such k-tuples can be found by partial sums of binomial coefficients. The other expressions give equivalent forms.

Our approach to proving Theorem 1 will be to "slice" the set of tuples by gathering them according to the last term. We can then enumerate each slice and add up the results to establish the theorem. To this end we will use the following lemma.

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Lemma 2. *Let $\mu(n)$ denote the Moebius function. Then*

$$\big|\{(a_1,a_2,\dots,a_\ell) : 1 \le a_1 \le a_2 \le \cdots \le a_\ell \le m,\ \gcd(a_1,a_2,\dots,a_\ell,m)=1\}\big| \\ = \sum_{d|m} \mu(m/d)\binom{d+\ell-1}{\ell}.$$

Proof. We first observe that

$$\big|\{(a_1,a_2,\dots,a_\ell) : 1 \le a_1 \le a_2 \le \cdots \le a_\ell \le m,\ \gcd(a_1,a_2,\dots,a_\ell,m)=1\}\big| = \\ \sum_{d|m} \mu(d)\big|\{(a_1,a_2,\dots,a_\ell) : 1 \le a_1 \le a_2 \le \cdots \le a_\ell \le m,\ d|\gcd(a_1,a_2,\dots,a_\ell,m)\}\big|.$$

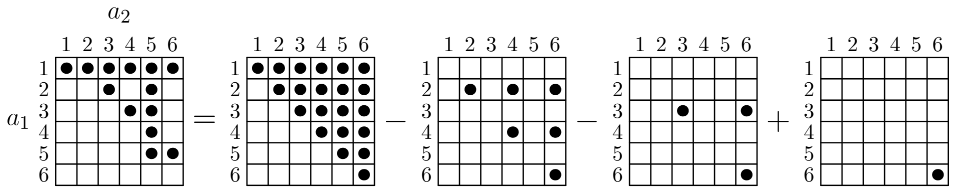
This follows by noting that the contribution of $(a_1,a_2,\dots,a_\ell)$ to the above sum will be $\sum_{e|f}\mu(e)$, where $f=\gcd(a_1,a_2,\dots,a_\ell)$. By the property of the Moebius function this will give a contribution of 1 when the $\gcd = 1$ and 0 when $\gcd > 1$, establishing the equality.

We also have that,

$$\big|\{(a_1,a_2,\dots,a_\ell) : 1 \le a_1 \le a_2 \le \cdots \le a_\ell \le m,\ d|\gcd(a_1,a_2,\dots,a_\ell,m)\}\big| \\ = \big|\{(a_1,a_2,\dots,a_\ell) : 1 \le a_1 \le a_2 \le \cdots \le a_\ell \le m/d\}\big| = \binom{m/d+\ell-1}{\ell}. \quad (1)$$

The first equality follows by the bijective map $(a_1,\dots,a_\ell)\leftrightarrow(a_1/d,\dots,a_\ell/d)$ between the two sets while the second is trivial (e.g., by induction). Combining the two above results and reindexing the sum concludes the proof of the lemma. ☐

A pictorial example of the lemma when $\ell = 2$ and $m = 6$ is shown below.



$$\left|\left\{\begin{matrix} a_1 \le a_2 \\ 1 = \gcd \end{matrix}\right\}\right| = \left|\left\{\begin{matrix} a_1 \le a_2 \\ 1\,|\gcd \end{matrix}\right\}\right| - \left|\left\{\begin{matrix} a_1 \le a_2 \\ 2\,|\gcd \end{matrix}\right\}\right| - \left|\left\{\begin{matrix} a_1 \le a_2 \\ 3\,|\gcd \end{matrix}\right\}\right| + \left|\left\{\begin{matrix} a_1 \le a_2 \\ 6\,|\gcd \end{matrix}\right\}\right|$$

Proof of Theorem 1. For the case $k = 1$ the left hand side is $\big|\{(1)\}\big| = 1$, while the right hand sides are

$$\sum_{i=1}^{n}\sum_{d|i}\mu(d) = \sum_{i\ge 1}\lfloor n/i\rfloor \mu(i) = \sum_{i\ge 1} M(\lfloor n/i\rfloor) = \sum_{i=1}^{n}[i=1] = 1.$$

(This is a well known result attributed to Lehman [1].)

For the case $k \geq 2$ we use the lemma to get the following.

$$\begin{aligned}
&\left|\{(a_1, a_2, \ldots, a_k) : 1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq n,\ \gcd(a_1, a_2, \ldots, a_k) = 1\}\right| \\
= &\sum_{m=1}^{n} \left|\{(a_1, a_2, \ldots, a_{k-1}) : 1 \leq a_1 \leq a_2 \leq \cdots \leq a_{k-1} \leq m,\ \gcd(a_1, a_2, \ldots, a_{k-1}, m) = 1\}\right| \\
= &\sum_{m=1}^{n} \sum_{d|m} \mu(m/d) \binom{d+k-2}{k-1}.
\end{aligned}$$

To get the other equivalent forms we simply group, i.e.,

$$\sum_{m=1}^{n} \sum_{d|m} \mu(m/d) \binom{d+k-2}{k-1} =$$

$$\sum_{i=1}^{n} \big(\mu(i/i) + \mu(2i/i) + \cdots + \mu(i\lfloor n/i \rfloor/i)\big) \binom{i+k-2}{k-1} = \sum_{i=1}^{n} M(\lfloor n/i \rfloor) \binom{i+k-2}{k-1},$$

and similarly,

$$\sum_{m=1}^{n} \sum_{d|m} \mu(m/d) \binom{d+k-2}{k-1} =$$

$$\sum_{i=1}^{n} \mu(i) \left(\binom{1+k-2}{k-1} + \binom{2+k-2}{k-1} + \cdots + \binom{\lfloor n/i \rfloor + k-2}{k-1} \right) = \sum_{i=1}^{n} \mu(i) \binom{\lfloor n/i \rfloor + k-1}{k}.$$

Concluding the proof. $\square$

References

[1] R. S. Lehman, “On Liouville’s function” *Math. Comp.* **14** 1960, 311–320.

[2] A. M. Odlyzko and H. J. J. te Riele, “Disproof of the Mertens conjecture”, J. Reine Angew. Math. **357** 1985, 138–160.

[3] N. J. A. Sloane, *On-line encyclopedia of integer sequences*, `www.research.att.com/~njas/sequences/`.