On permutations which are 1324 and $\overline{2143}$ avoiding

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Abstract

We consider permutations which are 1324 and $\overline{2143}$ avoiding, where $\overline{2143}$ avoiding means that it is 2143 avoiding with the additional Bruhat restriction $\{2 \leftrightarrow 3\}$. In particular, for every permutation π we will construct a linear map L_π and a labeled graph G_π and will show that the following three conditions are equivalent: π is 1324 and $\overline{2143}$ avoiding; L_π is onto; G_π is a forest.

We will also give some properties of G_π and show that the number of $\pi \in S_n$ whose graph is a path is $2^{n-1} - 1$.

1 Introduction

A permutation $\pi \in S_n$ contains the pattern 1324 if for some indices $p < q < r < s$ we have $\pi(p) < \pi(r) < \pi(q) < \pi(s)$. Similarly, we say that π contains the pattern $\overline{2143}$ if for some indices $p < q < r < s$ we have $\pi(q) < \pi(p) < \pi(s) < \pi(r)$ and there is no $p < t < s$ such that $\pi(p) < \pi(t) < \pi(s)$. When π does not contain the pattern 1324 and $\overline{2143}$ we say it is 1324 and $\overline{2143}$ avoiding.

We note that $\overline{2143}$ in the language of Bruhat restricted pattern avoidance is 2143 avoiding with Bruhat restriction $\{2{\leftrightarrow}3\}$. (For more information about Bruhat restricted order see [2, 9].) As an example, 21354 has the pattern 2143 but it is $\overline{2143}$ avoiding because the permutation does not have $\{2{\leftrightarrow}4\}$.

Our basic object of consideration will be the following construction. For $\pi \in S_n$ label n columns by $\pi(1), \pi(2), \ldots, \pi(n)$ and place vertical poles between the columns. For $i < j$ we draw a horizontal bar between column i and column j if and only if $\{i{\leftrightarrow}j\}$, that is, $\pi(i) < \pi(j)$ and there is no $i < k < j$ so that $\pi(i) < \pi(k) < \pi(j)$. (So the bar in $\overline{2143}$ indicates that in the diagram there is a bar between the columns

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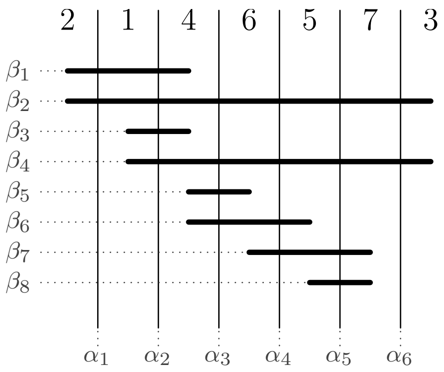
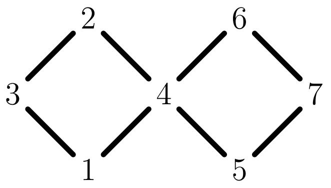
corresponding to the 2 and 3.) An example of this construction is shown below for $\pi = 2146573 \in S_7$.





We can use the construction to produce two objects. First we define a linear map by associating variables α_i with the vertical poles and β_j with the horizontal bars. If the jth horizontal bar starts in column s and goes to column t then we let

$$\beta_j = \sum_{i=s}^{t-1} \alpha_i. \tag{1}$$

This defines a map $L_\pi : \mathbb{Z}^{n-1} \to \mathbb{Z}^{\ell(\pi)}$ where $\ell(\pi)$ is the number of horizontal bars in the diagram. In the example above we have $\beta_1 = \alpha_1 + \alpha_2, \beta_2 = \alpha 1 + \cdots + \alpha_6, \ldots, \beta_8 = \alpha_5$.

The second object is a graph, G_π, which has as vertices the columns and there is an edge between two vertices if and only if there is a bar in the construction between the two columns. The graph associated with the permutation above is shown on the right.

Our main result is the following theorem.

Theorem 1. *For $\pi \in S_n$ the following are equivalent.*

(1) π is 1324 and $\overline{2143}$ avoiding;

(2) L_π is onto;

(3) G_π contains no cycles.

We also have the following related result.

Corollary 2. *The map L_π is a bijection if and only if G_π is a tree.*

We will proceed as follows, in the remaining part of the introduction we will mention a connection with Schubert varieties. In the next section we will introduce a sorting on the horizontal bars and use this to prove Theorem 1. We will then briefly mention some upper and lower bounds for the number of such permutations. Finally in the last part of the paper we will discuss some of the graphical properties of G_π.

1.1 Schubert varieties and 1324 and $\overline{2143}$ avoiding permutations

The original motivation for studying 1324 and $\overline{2143}$ came from a question of Woo and Yong related to Schubert varieties. It is known that Schubert varieties can be indexed by permutations [5]. Various properties of permutations have been translated into properties of Schubert varieties, one famous example is a variety is smooth if the permutation avoids the pattern 1324 and 2143 [6].

A weakening of smoothness if factoriality, an algebra-geometric condition which states that all local rings are unique factorization domains. Cortez [3] showed that 1324 and $\overline{2143}$ avoidance is a necessary and *generically* sufficient to characterize which Schubert varieties have factoriality. Here generic has the following sense, the variety is smooth at almost all points of the Schubert variety X_π but has a closed subset Y_π where it is not smooth, and in that closed subset it is factorial at *almost* all points.

Woo and Yong [11] established the following.

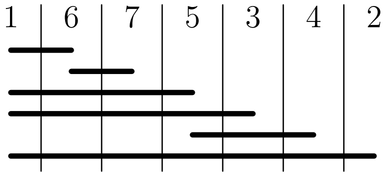
Proposition 3. *The Schubert variety indexed by π has factoriality if and only if L_π is onto.*

Woo and Yong then conjectured that L_π is onto if and only if it is 1324 and $\overline{2143}$ avoiding (a strengthening of the result of Cortez). The genesis of the paper was in proving this conjecture of Woo and Yong. For more information between the construction of the diagram and the connection to geometry the reader is referred to [10].

2 Sorting the horizontal bars

In the construction of the diagram for a permutation we placed no condition on the ordering of the horizontal bars from top to bottom, we now describe a way to sort them. Add a second diagram with the same columns but no horizontal bars. We now look for bars to move to the second diagram by scanning the vertical poles looking for any pole which is intersected by exactly one horizontal bar. If we find such an intersection we move the corresponding horizontal bar to the second diagram and put it on top of any previously moved bars. We now repeat this process until either all the horizontal bars are moved over or no pole intersects exactly one horizontal bar.

An example for what the second diagram (which we call the sorted diagram) might look like for the permutation $1675342 \in S_7$ is shown below (note in this case we were able to move over all the bars).



In terms of the equations in (1), the above sorting is arranging the system of equations so that it can be solved by backward substitution in going from the top equation to the bottom equation. So if we have been able to fully sort (as in the example above) then for an assignment of integers to the horizontal bars we assign values to the vertical poles in the following way.

Starting with the top horizontal bar assign integer(s) to the vertical pole(s) it intersects to produce the correct value. We then move down through the horizontal bars assigning values to poles as needed. The key is that by our sorting, as we move down each horizontal bar will intersect a vertical pole that has not yet been assigned a value, and so we can guarantee an assignment of the vertical poles to get the desired values on the horizontal bars. This establishes the following.

Lemma 4. *If we are able to fully sort the bars then* L_π *is onto.*

On the other hand, we have the following.

Lemma 5. *If we are not able to sort the bars then* π *contains* 1324 *or* $\overline{2143}$.

Proof. If we stopped before all the bars have been moved over then it must be the case that for what remains all the vertical poles intersect either zero, or two or more horizontal bars. We will work with these remaining (i.e., unmoved) horizontal bars.

Suppose that column a is the leftmost column which has the start of a bar, then as noted above it must be the start of at least two bars (otherwise we would have moved the bar over). Let c denote the column where the *longest* horizontal bar starting in column a ends. Let b be the rightmost column satisfying $a < b < c$ and $\pi(c) < \pi(b)$ (such a b exists because the end of a second bar that starts in a has this property).

We now consider cases on how to cover the vertical bar to the right of column b with a second horizontal bar.

Case (1) There is a horizontal bar that begins at b and goes beyond c to some position d. In this case we have that $a < b < c < d$ while $\pi(a) < \pi(c) < \pi(b) < \pi(d)$ and so π has the pattern 1324.

Case (2) There is a horizontal bar that begins at column d where $d < b$ and crosses to some column e where $e > b$. Since d lies between a and c we must have that $\pi(d) < \pi(a)$ or $\pi(d) > \pi(c)$. So we consider subcases.

Case (2a) If $\pi(d) < \pi(a)$ then we have that $a < d < b < c$ and $\pi(d) < \pi(a) < \pi(c) < \pi(b)$ and since there is a horizontal bar from a to c, π has the pattern $\overline{2143}$. (Note this includes the possibility that $c = e$.)

Case (2bi) Suppose that $\pi(b) < \pi(d)$. Then we note that we have $a < d < b < e$ and $\pi(a) < \pi(b) < \pi(d) < \pi(e)$ and so π has the pattern 1324.

Case (2bii) Suppose that $\pi(c) < \pi(d) < \pi(b)$, note by assumption we have $c < e$. Then we note that we have $a < d < c < e$ and $\pi(a) < \pi(c) < \pi(d) < \pi(e)$ and so π has the pattern 1324. $\square$

By taking the contrapositive of Lemma 5 with Lemma 4 we have that (1)⇒(2) in Theorem 1. To get that (1)⇒(3) suppose that G_π contained a cycle. Then if we only consider the bars that correspond to the edges of the cycle they cannot be sorted as outlined above. (To see this, if they could be sorted then some pole would intersect only one bar, by removing the edge that corresponds to the bar we would disconnect the cycle, which is impossible.) In particular, we cannot sort the full set of bars and so π contains 1342 and $\overline{2143}$. Now taking the contrapositive gives the result.

2.1 Structure inside 1324 and $\overline{2143}$

To prove the remainder of Theorem 1 we will look at the structure found in permutations containing 1324 and $\overline{2143}$. We will use the following lemmas.

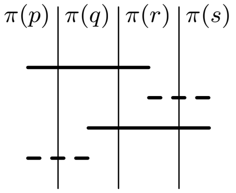
Lemma 6. *If $p < q$ and $\pi(p) < \pi(q)$ then there is a sequence $p = p_0 < p_1 < \cdots < p_k = q$ such that $\pi(p_i) < \pi(p_{i+1})$ and in the diagram for π there are horizontal bars from column p_i to column p_{i+1} for each $i = 0, \ldots, k-1$.*

Lemma 7. *Given a permutation π,*

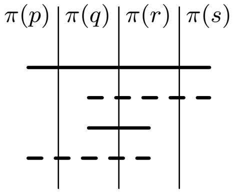
- *(a) if π contains the pattern $\overline{2143}$ then there are indices $p < q < r < s$ such that $\pi(q) < \pi(p) < \pi(s) < \pi(r)$ and in the diagram for π there are horizontal bars from p to s and from q to r.*
- *(b) if π contains the pattern 1324 then there are indices $p < q < r < s$ such that $\pi(p) < \pi(r) < \pi(q) < \pi(s)$ and in the diagram for π there are horizontal bars from p to r and from q to s.*

The proof of Lemma 6 follows from the rule for placing horizontal bars. The proof of Lemma 7 follows by noting that if the we do not already have the requisite bars then we can continue to tighten the pattern until it does. We are now ready to finish the proof of Theorem 1. The intuition is contained in the following pictures, whose structures follow from Lemmas 6 and 7. The solid lines indicate a single bar and the dashed lines indicate a sequence of bars.

1324 case



$\overline{2143}$ case



From the pictures we can read off cycles in G_π. For example in the 1324 case we have a cycle that starts at $\pi(p)$ goes to $\pi(r)$ then by a sequence of edges goes to $\pi(s)$ then to $\pi(q)$ and finally by another sequence of edges we return to $\pi(p)$. So if the permutation contains 1324 or $\overline{2143}$ then G_π contains a cycle. Taking the contrapositive gives (3)⇒(1) in Theorem 1.

Finally, we also see that we are forced to have nontrivial linear dependencies among the β_i. In both cases the sum the solid bars equals the sum of the dashed bars. For example, in the 1324 case these sums are both equal to the sum of the variables corresponding to the poles between p and s. So if the permutation contains 1324 or $\overline{2143}$ then L_π is not onto. Taking the contrapositive gives (2)⇒(1) in Theorem 1. This concludes the proof of the theorem.

For Corollary 2 we note that the map must be onto and so the graph not have a cycle. In order to be into we need to have at least $n-1$ bars (i.e., the same as the number of poles). In terms of G_π this means that it is an acyclic graph with at least $n-1$ bars, in other words a tree. We can even be more precise, if we look at the results of the sorting when we "revealed" the bars going from top to bottom each bar was associated with at least one pole (i.e., the previously unused pole(s)). If we have $n-1$ bars then we must have that each bar is associated with exactly one pole. So the inverse is unique showing 1-1.

3 Bounds on avoiding permutations

The enumeration of permutations avoiding certain patterns has been an active area of study. The enumeration of permutations which are 1324 and $\overline{2143}$ avoiding has not yet been researched but we can use some work which has been done in other patterns to get upper and lower bounds. For instance the number of permutations which are 1324 avoiding has been studied [7] and provides an upper bound. Also the number of permutations which are 1324 and 2143 avoiding had been studied [1] and provides a lower bound.

We compare the initial terms of these three avoidance patterns. Of course much work remains to be done in generating the number of permutations which are 1324 and $\overline{2143}$ avoiding for general n.

n	1324 and 2143 avoiding	1324 and $\overline{2143}$ avoiding	1324 avoiding
1	1	1	1
2	2	2	2
3	6	6	6
4	22	22	23
5	88	89	103
6	366	379	513
7	1552	1661	2762
8	6652	7405	15793
9	28696	33367	94776
10	124310	151398	591950
11	540040	690147	3824112
12	2350820	3156112	25431452
13	10248248	14465746	173453058
14	44725516	66409493	1209639642
15	195354368	305232025	8604450011
16	853829272	1404129530	62300851632

If we enumerate the permutations where G_π is a tree then we get the sequence starting 1, 1, 3, 11, 44, 184, 789, 3435, 15100, 66806, 296870, 1323318, 5911972, 26455294, 118528793. The start of this sequence (at least as far as the author was able to carry it) agrees with a known integer sequence, namely, the number of stacked directed animals on a triangular lattice [8]. It would be interesting to now if there was a bijective correspondence between such objects.

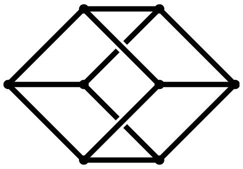
4 More on the graphical interpretation of a permutation

While the initial investigation into permutations which avoid 1324 and $\overline{2143}$ focused on the linear map, the graphical interpretation has many interesting avenues for investigation. The reader might enjoy trying to prove the following two simple results.

Lemma 8. *G_π is isomorphic to $G_{\pi^{-1}}$.*

Lemma 9. *G_π is triangle free. In particular, $\ell(\pi) \leq \lfloor n^2/4 \rfloor$ and this bound is sharp.*

Lemma 9 gives a necessary condition for a graph be realizable by a permutation. However, it is not a sufficient condition, for example the triangle-free graph below has no permutation in S_8 which produces the graph.



An interesting problem would be to give a characterization of graphs which are isomorphic to G_π for some π. We note that all forests are realizable by some π, by the following construction: Take any forest on n vertices and add an extra vertex, which we call the root vertex. Connect a vertex from every component in the forest to the root vertex. Starting at the root vertex label the root 0 and walk around the outside of the tree labeling the vertices in order of first visit (in other words labeling by a depth first search). Finally remove the root vertex, the resulting graph is a G_π.

To recover the actual permutation there are several unranking algorithms, all based on Lemma 6, which show that the map $\pi \to G_\pi$ is one-to-one. One such algorithm is as follows. Starting with the labeled graph we orient the edges so that the edges go from low-labeled vertices to high-labeled vertices. We then unrank π entry by entry starting at the last entry and going backwards in the following manner: Find the vertex with lowest index in the graph which has outdegree 0, put that index into the rightmost available slot in the permutation, remove the corresponding vertex and incident edges from the graph, and repeat.

4.1 Permutations which produce paths

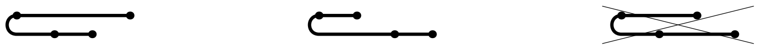
We conclude with one more type of problem that can be explored relating permutations to graphs. If we count the number of permutations for which G_π is a path then we get the sequence 1, 1, 3, 7, 15, 31, 63, 127 and so on. This has an obvious pattern and indeed we have the following result.

Theorem 10. *For $n \geq 2$ there are $2^{n-1} - 1$ permutations $\pi \in S_n$ for which G_π is a path.*

To prove the theorem we give a bijection between numbers between 1 and $2^{n-1} - 1$ and permutations for which G_π is a graph. A quick outline of the bijection is as follows. We start with a number between 1 and $2^{n-1} - 1$ and expand it in binary (include leading zeros so that we have $n - 1$ terms). The 0s and 1s in the binary expansion are going to code the relative order of the terms in the path. That is starting at the end vertex with lowest index and then moving to the other end vertex a 0 will denote a decrease in the index and a 1 will denote an increase in the index. From these relationships we will be able to reconstruct the diagram for the permutation and then the actual permutation. The reverse map is easy.

We demonstrate with an example. Suppose $n = 10$ and we want to find the permutation corresponding to 46 then we write $46 = (000101110)_2$. To emphasize the relationships we rewrite this as $46 = \downarrow\downarrow\downarrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow$. We now interlace the arrows with an alphabet, and get the following, `a↓b↓c↓d↑e↓f↑g↑h↑i↓j`. The arrows are encoding the relative order of the terms, so we have that `a>b>c>d<e>f<g<h<i>j`. We will also add one more implicit assumption that `a<j`.

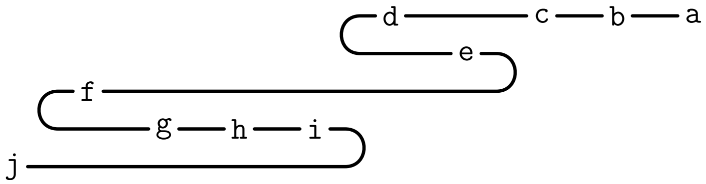
The next step is to get the relative order of the alphabet in the permutation. This is accomplished by translating the arrows. For a down-arrow we move to the left for an up-arrow we move to the right. The question is what happens when we change direction. In that case we have one of three options: an in-move, on out-move, or interlace. An example of each is shown below for $\downarrow\uparrow\uparrow$.



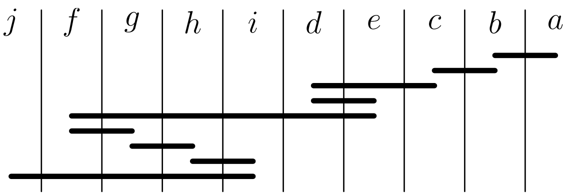
in-move out-move interlace

For an in-move we turn and immediately start going and stay together, in an out-move we turn and go as far as we can and then start going, in an interlacing move we interlace. It is easy to show that if we interlace we cannot get a permutation that produces a path because additional edges will be forced on us. So we are restricted to in- and out-moves. It is also easy to show that we must alternate in- and out-moves, i.e., if we have two in-moves in a row then again we would be forced to have additional edges.

We will adopt the following conventions; when going from down-arrows to up-arrows we will do an in-move, when going from up-arrows to down-arrows we will do an out-move. With this convention it can be shown that as long as we do not have all down-arrows (which would correspond to the number 0) then in the labeling the first term will be smaller than the last when we have found the permutation (i.e., `a<j` in our example). Using this we now have the following picture.



Which translates to the diagram below.



Finally we recover the permutation by the following procedure. Start at the right most column which has not yet been assigned. Count the number of columns that can be reached by moving left along horizontal bars, if that number is i put the $(i+1)$st unused number in that slot and repeat.

So for instance in our picture starting with the column labeled `a` we can get to three columns, namely `b,c,d`. So we put a 4 in for `a`. Then for column `b` we will get a 3, for column `c` a 2, for column `e` we can reach two columns, namely `d,f` and so we will put a 6 (the current third unused number). Repeating this throughout we get the desired permutation: 9 5 7 8 10 1 6 2 3 4.

References

[1] M. Bóna, "The permutation classes equinumerous to the smooth class", Electron. J. Combin. **5** (1998), no. 1, Research Paper 31, 12 pp.

[2] M. Bóna, *Combinatorics of Permutations*, CRC, Boca Raton, 2004.

[3] A. Cortez, "Singularités génériques et quasi-résolutions des variétés de Schubert pour le groupe linéaire", Adv. Math. **178** (2003) p. 396–445.

[4] S. Jukna, *Extremal Combinatorics*, Springer, Berlin, 2001.

[5] W. Fulton, Young Tableux, Cambridge University Press, Cambridge, 1997.

[6] V. Lakshmibai and B. Sanhya, "Criterion for smoothness of Schubert varieties in Sl$(n)/B$", Proc. Indian Acad. Sci. Math. Sci., **100** (1990) p. 45–52

[7] D. Marinov and R. Radoičić, "Generating trees of (reducible) 1324-avoiding permutations", Electron. J. Combin. **9** (2002/03), no. 2, Research paper 13, 9 pp.

[8] M. Bousquet-Mélou and A. Rechnitzer, "Lattice animals and heaps of dimers", Discrete Math. **258** (2002) p. 235–274.

[9] R. Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge University Press, Cambridge, 1999.

[10] A. Woo and A. Yong, “When is a Schubert variety Gorenstein?”, preprint.

[11] A. Woo and A. Yong, personal communication.