

Homework 0 solutions

- (1) We have the following using separation of variables:

$$\begin{aligned}\frac{dy}{dx} &= y^2 e^{3x} \\ \frac{dy}{y^2} &= e^{3x} dx \\ \int \frac{dy}{y^2} &= \int e^{3x} dx \\ \frac{-1}{y} &= \frac{1}{3} e^{3x} + C \\ y &= \frac{-1}{\frac{1}{3} e^{3x} + C} \quad \text{or} \quad \frac{-3}{e^{3x} + D}\end{aligned}$$

Note: many people made mistakes in going between the last two lines. For instance they would flip each piece on the right side individually and get something like $3e^{-3x} + C$. Be careful with the algebra!!

- (2) Recall that to find the integrating factor we are looking for some function u (as a function of x) so that

$$uy' + \frac{u}{x}y = uy' + u'y = (uy)',$$

so in particular we need to have $u/x = u'$. It is easy to see that such a u in this case is $u = x$. (Note in general if we have $y + f(x)y' = g(x)$ then the integrating factor is $e^{\int f(x) dx}$, in our case we have $e^{\int 1/x dx} = e^{\ln x} = x$.) Using the integrating factor we then have the following:

$$\begin{aligned}y' + \frac{y}{x} &= x^2 + 1 \\ xy' + y &= x^3 + x \\ (xy)' &= x^3 + x \\ xy &= \int (x^3 + x) dx = \frac{1}{4}x^4 + \frac{1}{2}x^2 + C \\ y &= \frac{1}{4}x^3 + \frac{1}{2}x + \frac{C}{x}\end{aligned}$$

- (3) Given that $y'' - 12y' + 35y = 0$ we have that the characteristic equation is

$$r^2 - 12r + 35 = (r - 5)(r - 7) = 0.$$

(The idea being that we are guessing that y has the form $y = e^{rx}$ and then are determining which value(s) of r satisfy the differential equation.) Since we have two

roots to this linear differential equation then by superposition we have that the general solution is

$$y = c_1 e^{5x} + c_2 e^{7x}.$$

- (4) In solving problems of this type there are three general steps: (i) solve the homogeneous problem; (ii) solve the nonhomogeneous problem, i.e., find a particular solution; (iii) take care of the initial conditions.

For the homogeneous problem we consider $y'' - 3y' + 2y = 0$ which, using the same methods as problem (2), has characteristic equation $r^2 - 3r + 2 = (r - 1)(r - 2) = 0$ so that the homogeneous solution is $y_H = c_1 e^x + c_2 e^{2x}$.

For the nonhomogeneous part we see that the function $\sin x$ is not part of the homogeneous solution (a good thing!) and so using the method of undetermined coefficients we make the guess

$$y = A \sin x + B \cos x.$$

Substituting this and its derivatives then $y'' - 3y' + 2y = 2 \sin x$ becomes

$$(-A \sin x - B \cos x) - 3(A \cos x - B \sin x) + 2(A \sin x + B \cos x) = 2 \sin x.$$

Gathering the coefficients we then have

$$(-A + 3B + 2A) \sin x + (-B - 3A + 2B) \cos x = 2 \sin x,$$

which reduces to the set of equations

$$\begin{aligned} A + 3B &= 2 \\ -3A + B &= 0 \end{aligned}$$

Then substituting $B = 3A$ into the first equation we have that $10A = 2$ so that $A = 1/5$ and $B = 3/5$ giving $y_P = (1/5) \sin x + (3/5) \cos x$.

Combining the two solutions above the general solution and its derivative is then

$$\begin{aligned} y &= c_1 e^x + c_2 e^{2x} + \frac{1}{5} \sin x + \frac{3}{5} \cos x, \\ y' &= c_1 e^x + 2c_2 e^{2x} + \frac{1}{5} \cos x - \frac{3}{5} \sin x, \end{aligned}$$

Using the initial conditions $y(0) = 1$ and $y'(0) = 0$ we then have the set of equations

$$\begin{aligned} 1 &= c_1 + c_2 + \frac{3}{5} \\ 0 &= c_1 + 2c_2 + \frac{1}{5} \end{aligned}$$

which can easily be solved again by substitution or using Gaussian elimination to give $c_1 = 1$ and $c_2 = -3/5$ giving us the final answer

$$y = e^x - \frac{3}{5} e^{2x} + \frac{1}{5} \sin x + \frac{3}{5} \cos x.$$

- (5) Recall that λ is an eigenvalue of a matrix A if and only if $\det(A - \lambda I) = 0$. So we consider

$$0 = \det(A - \lambda I) = \det \left(\begin{bmatrix} 1 - \lambda & 2 \\ 3 & 2 - \lambda \end{bmatrix} \right) = (1 - \lambda)(2 - \lambda) - (3)(2) = \lambda^2 - 3\lambda - 4.$$

Since the right hand side factors as $(\lambda + 1)(\lambda - 4)$ we conclude that the eigenvalues are -1 and 4 .

To find the eigenvectors we recall that \mathbf{x} is an eigenvector if and only if $(A - \lambda I)\mathbf{x} = \mathbf{0}$. So for $\lambda = -1$ we have

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which reduces to the equations

$$\begin{aligned} 2a + 2b &= 0, \\ 3a + 3b &= 0. \end{aligned}$$

These are essentially the same equation (as we should expect) so that as long as $a = -b$ the relationship is satisfied. So we have that for $\lambda = -1$ the eigenvector is any of the form

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -a \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

for any $a \neq 0$.

Similarly if we consider the value $\lambda = 4$ we have

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which will similarly be satisfied as long as $3a = 2b$. So we have that for $\lambda = 4$ the eigenvector is any of the form

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ (3/2)a \end{bmatrix} = a \begin{bmatrix} 1 \\ 3/2 \end{bmatrix},$$

for any $a \neq 0$.