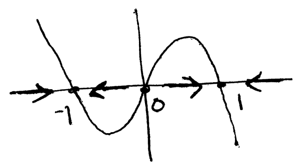
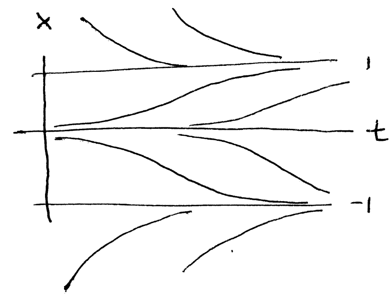
Homework 1 solutions

2.2.3 For $\dot{x} = x - x^3$ it is easy to see that the fixed points are -1, 0 and 1 (which are respectively stable, unstable and stable). The sketch of the vector field and some $x(t)$ for different initial conditions are shown below.





If we now attempt to find an explicit solution for $\dot{x} = x - x^3$ we have

$$dt = \frac{dx}{x - x^3} = \frac{dx}{x(1-x)(1+x)} = \frac{dx}{x} + \frac{1}{2}\frac{dx}{1-x} - \frac{1}{2}\frac{dx}{1+x}.$$

[The last step being done by partial fractions.] If we now integrate both sides and multiply by 2 we have

$$2t + C = 2\ln x - \ln(1-x) - \ln(1+x) = \ln\left(\frac{x^2}{1-x^2}\right).$$

[The last step using properties of natural logarithms.] In particular by using the exponential function we now have

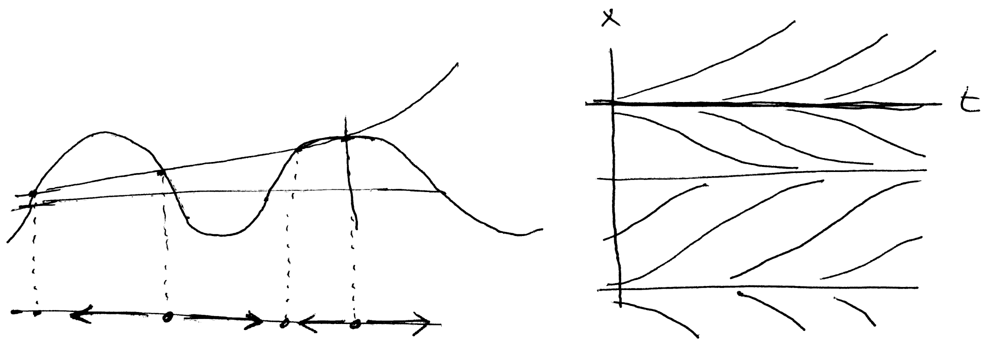
$$De^{2t} = \frac{x^2}{1-x^2} \qquad \text{or upon rearranging} \qquad x^2 = \frac{De^{2t}}{1+De^{2t}} = \frac{1}{de^{-2t}+1}.$$

[The last step is valid for $D \neq 0$, which corresponds to the fixed point $x = 0$.] So we have that

$$x = \pm\sqrt{\frac{1}{de^{-2t}+1}} \qquad \text{or} \qquad x = 0.$$

Note that as $t \to \infty$ that for $x \neq 0$ the above solutions head to either 1 or -1 (depending on the sign in front of the square root). This agrees with our initial analysis related to the stability of the fixed points.

2.2.7 For $\dot{x} = e^x - \cos x$ we plot the two curves e^x and $\cos x$ and look for intersections. The first intersection occurs at $x = 0$ and corresponds to an unstable point. There are infinitely other intersection points and they will oscillate between being stable and unstable as shown below on the left. Some $x(t)$ for different initial conditions is shown below on the left.



Attempting to integrate, it does not take long to see that we will not make much progress, so we will have to content ourselves only with understanding the qualitative behavior.

2.2.10 a) For $\dot{x} = f(x)$ to have every real number a fixed point we need $f(x)$ to be zero for all real x. The only function which does that is $f(x) = 0$, giving $\dot{x} = 0$ (i.e., x is a constant).

b) Again we must find a function which is zero only at integers. There are several possibilities, the simplest being perhaps $\sin(\pi x)$, giving $\dot{x} = \sin(\pi x)$.

c) If we had three consecutive fixed points and the outer two were stable then the vector field would flow away from the middle fixed point and that would have to be *un*stable. So there is no example of an $f(x)$ satisfying this.

d) To have no fixed points we need to have a function which is never zero. There are plenty of examples, one of the simplest being $\dot{x} = 1$.

e) To have 100 fixed points we need to have a function which is 0 for precisely 100 values of x. One way to do this is to make a polynomial that is zero at 100 places, for instance, $\dot{x} = x(x-1)(x-2)\cdots(x-99)$.

2.3.1 a) (Separation of variables approach)

$$\dot{N} = rN\big(1 - \frac{N}{K}\big) = \frac{r}{K}N(K-N) \qquad \text{becomes} \qquad \frac{dN}{N(K-N)} = \frac{r}{K}dt.$$

We now work on the left hand side, note that in order for

$$\frac{1}{N(K-N)} = \frac{A}{N} + \frac{B}{K-N} \qquad \text{we need} \qquad 1 = A(K-N) + BN,$$

since this last relationship should hold for any value of N choosing $N = 0$ gives $A = 1/K$ and $N = K$ gives $B = 1/K$. Using this we now have

$$\frac{1}{K}\frac{dN}{N} + \frac{1}{K}\frac{dN}{K-N} = \frac{r}{K}dt \qquad \text{or} \qquad \frac{dN}{N} + \frac{dN}{K-N} = r\,dt.$$

We now are ready to integrate to get

$$\ln N - \ln(K-N) = \ln\left(\frac{N}{K-N}\right) = rt + C \qquad \text{so} \qquad \frac{N}{K-N} = De^{rt}.$$

We can now rearrange this to solve for N and get that N has the following form

$$N = \frac{K}{1+de^{-rt}} \qquad \text{or} \qquad N = 0.$$

[The latter case corresponding to the situation $D = 0$.] Now using our initial condition $N(0) = N_0$ we can solve for d and we get that

$$N = \frac{K}{1+(K/N_0 - 1)e^{-rt}} \quad (N_0 \neq 0) \qquad \text{or} \qquad N = 0 \quad (N_0 = 0).$$

b) (Change of variables approach)
If we let $x = 1/N$ then $N = 1/x$ and $\dot{N} = -\dot{x}/x^2$. By replacing every occurrence of N and $\dot{N}$ in terms of x and $\dot{x}$

$$\dot{N} = rN(1 - N/K) \qquad \text{becomes} \qquad -\frac{\dot{x}}{x^2} = r\frac{1}{x}\big(1 - \frac{1}{xK}\big).$$

Multiplying both sides by -x^2 and simplifying we then get

$$\dot{x} = -r\big(x - \frac{1}{K}\big) \qquad \text{or} \qquad \frac{dx}{\left(x - \frac{1}{K}\right)} = -r\,dt.$$

Integrating both sides then gives

$$\ln\big(x - \frac{1}{K}\big) = -rt + C \qquad \text{or} \qquad x - \frac{1}{K} = De^{-rt} \qquad \text{or} \qquad x = \frac{1+de^{-rt}}{K}.$$

Since $N = 1/x$ we can now conclude that

$$N = \frac{K}{1+de^{-rt}} \qquad \text{or} \qquad N = 0.$$

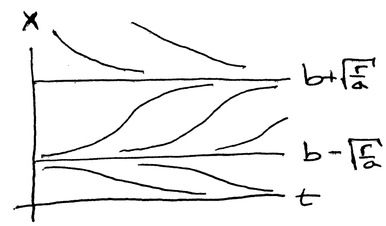
Then we can again use initial conditions like as in the previous part to get

$$N = \frac{K}{1+(K/N_0 - 1)e^{-rt}} \quad (N_0 \neq 0) \qquad \text{or} \qquad N = 0 \quad (N_0 = 0).$$

2.3.4 a) For $\dot{N}/N = r - a(N-b)^2$, the right hand side is a parabola with its vertex located at $b = N$. In order for growth rate to me maximized we need the parabola to be going down (i.e., $a > 0$) and need $r > 0$ to ensure that there is growth (note in this context r corresponds to the maximum rate). Now considering $\dot{N} = N\big(r - a(N-b)^2\big)$ the fixed points will be at 0, $b - \sqrt{r/a}$ and $b + \sqrt{r/a}$ and further b lies intermediate between the last two values. [I will assume that $b - \sqrt{r/a} > 0$, so that we can compare the different fixed points.]

b) As mentioned above the fixed points are 0, $b - \sqrt{r/a}$ and $b + \sqrt{r/a}$. Simple analysis shows that 0 and $b + \sqrt{r/a}$ are stable while $b - \sqrt{r/a}$ is unstable.

c) A sketch of some of the solutions is shown below.



d) Some of the solutions are similar to the logistic model. However when $0 < b - \sqrt{r/a}$ (as done above) we see that there is a slight difference. In the logistic model as long as we start with any size population it will move to the carrying capacity. In this model we will move to the carrying capacity only if our population is at least $b - \sqrt{r/a}$ (i.e., we have some minimal threshold size), otherwise if our population is too small it will die out and go to 0 (different from the logistic model).

2.4.2 The fixed points of $\dot{x} = x(1-x)(2-x) = 2x - 3x^2 + x^3$ are $x = 0$, 1 and 2. Taking the derivative of $f(x) = 2x - 3x^2 + x^3$ we get $f'(x) = 2 - 6x + 3x^2$. We now have $f'(0) = 2 > 0$, $f'(1) = -1 < 0$ and $f'(2) = 2 > 0$ showing that 0 and 2 are unstable and 1 is stable.

2.4.8 First we note that the only fixed point of $\dot{N} = -aN\ln(bN) = f(N)$ is at $N = 1/b$ (since $\ln 1 = 0$), 0 does not qualify as a fixed point since $f(0)$ is undefined (i.e., is $0{\cdot}\infty$). Since $f'(N) = -a\ln(bN) - a$, we have that $f'(1/b) = -a < 0$ and so $1/b$ is stable.

2.5.1 a) Since we are going to be only considering the positive real numbers we see that for *any* c and $x > 0$ that $\dot{x} = -x^c < 0$ so that all points move to 0. In particular we have that 0 is stable for all c.

We say that x_0 is a fixed point of $\dot{x} = f(x)$ if $f(x_0) = 0$ (see the definition in the middle of page 19). For $f(x) = -x^c$ we see that for $c > 0$ that $f(0) = 0$. For $c = 0$ we have $f(0) = -0^0$ which is undefined and for $c < 0$ we have that $f(0) = -1/0^{-c}$ (i.e., involves a division by 0) which is again undefined.

Therefore the only time that 0 is a stable fixed point is when $c > 0$.

b) We can interpret this by saying that $\dot{x} = -x^c$, $x(0) = 1$ and we are trying to determine a finite t so that $x(t) = 0$. Since the equation is separable it can be rewritten as $x^{-c}\,dx = -dt$. Then integrating both sides we get the following

$$\begin{cases} \dfrac{x^{-c+1}}{-c+1} = -t + C & \text{if } c \neq 1; \\ \ln x = -t + C & \text{if } c = 1. \end{cases}$$

For the case that $c = 1$ we have (using the initial condition) that $x = e^{-t}$, for which it is well known that this function is never 0 so this will not reach 0 in finite time.

Now suppose that $c \neq 1$ then plugging in the initial condition $x(0) = 1$ and solving for C we have

$$\frac{x^{-c+1}}{-c+1} = -t + \frac{1}{-c+1}.$$

Now suppose that $c < 1$ (i.e., so $1 - c > 0$), then to find the time that it hits the origin we let $x = 0$ and solve for t and conclude that it hits the origin at $t = 1/(-c+1)$.

For $c > 1$ we have two approaches. The first approach is to rearrange the above expression to solve for t for which we have

$$t = \frac{x^{-c+1} - 1}{c - 1}.$$

Now to find when $x = 0$ we see that the above expression is undefined, but if we think of it as a limit trying to get closer to 0 then we have

$$t = \lim_{x \to 0} \frac{x^{-c+1} - 1}{c - 1} = \infty.$$

Showing that it takes infinitely long to get to the origin.

The second approach is to notice that for $c > 1$ we have that $\dot{x} = -x^c > -x$ (here we are also using the fact that since we start at 1 and are going towards 0 we can assume $0 \leq x \leq 1$). Since we know that the solution to $\dot{x} = -x$ never gets to the origin in finite time and since the solution $\dot{x} = -x^c$ moves to the origin even more *slowly* (this is what $\dot{x} = -x^c > -x$ is saying) then it also can never get to the origin in finite time.

So in conclusion, for $c \geq 1$ it will never get to the origin in finite time. For $c < 1$ it will hit the origin at time $1/(-c+1)$.

2.5.3 There are two methods that can be applied. Namely an analytic solution and comparison. We do both.

[Analytic solution.]

Starting with $\dot{x} = rx + x^3$ we see this is separable and upon rearranging we have

$$\frac{dx}{x(r + x^2)} = dt.$$

Doing partial fractions on the left we see that

$$\frac{1}{x(r+x^2)} = \frac{A}{x} + \frac{Bx+C}{r+x^2} \qquad \text{if} \qquad 1 = A(r+x^2) + (Bx+C)x.$$

Grouping coefficient we see that we need $Ar = 1$, $C = 0$ and $A + B = 0$. From this it follows that $A = 1/r$, $B = -1/r$ and $C = 0$. If we substitute this in and multiply both sides by r we have

$$\left(\frac{1}{x} - \frac{x}{r+x^2}\right) dx = r\,dt \qquad \text{so} \qquad \int \left(\frac{1}{x} - \frac{x}{r+x^2}\right) dx = \int r\,dt.$$

This gives us

$$\ln x - \frac{1}{2}\ln(r+x^2) = rt + C \qquad \text{or} \qquad \ln\left(\frac{x^2}{r+x^2}\right) = 2rt + C'.$$

Taking the exponent of both sides we are left with

$$\frac{x^2}{r+x^2} = C''e^{2rt} \qquad \text{or rearranging} \qquad x^2 = \frac{r}{De^{-2rt}-1}.$$

[In this last step we used that $C'' \neq 0$, the case $C'' = 0$ corresponds to the stable point of 0 which we are assuming $x_0 \neq 0$.] Plugging in the initial condition $x(0) = x_0$ we can see that $D = 1 + r/x_0^2$. Substituting this in and taking the square root we have that

$$x = \pm\sqrt{\frac{r}{(1+\frac{r}{x_0^2})e^{-2rt} - 1}}.$$

The sign is dependent on whether x_0 is positive or negative. Looking at the analytical solution we see that initially the denominator term is positive and as t increases the denominator term $\left(1+\frac{r}{x_0^2}\right)e^{-2rt} - 1$ gets closer to 0 (causing $x \to \pm\infty$ depending on the sign in front of the square root). In particular at

$$t = \frac{\ln\left(1+\frac{r}{x_0^2}\right)}{2r}$$

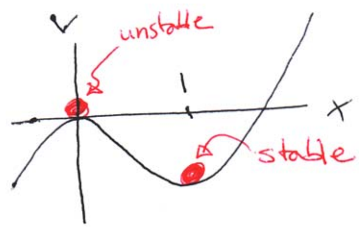
The denominator is 0 and this is a vertical asymptote for the solution.

[Comparison.]
We know from the example in the book that $\dot{x} = 1 + x^2$ has a vertical asymptote (i.e., goes to ∞ in finite time). So if we can show that *eventually* our solution to the differential equation must grow at least this fast then we would have that it must also blow up. In what follows we will assume that $x_0 > 0$, the same argument works for $x_0 < 0$ by just changing signs in all the appropriate places.

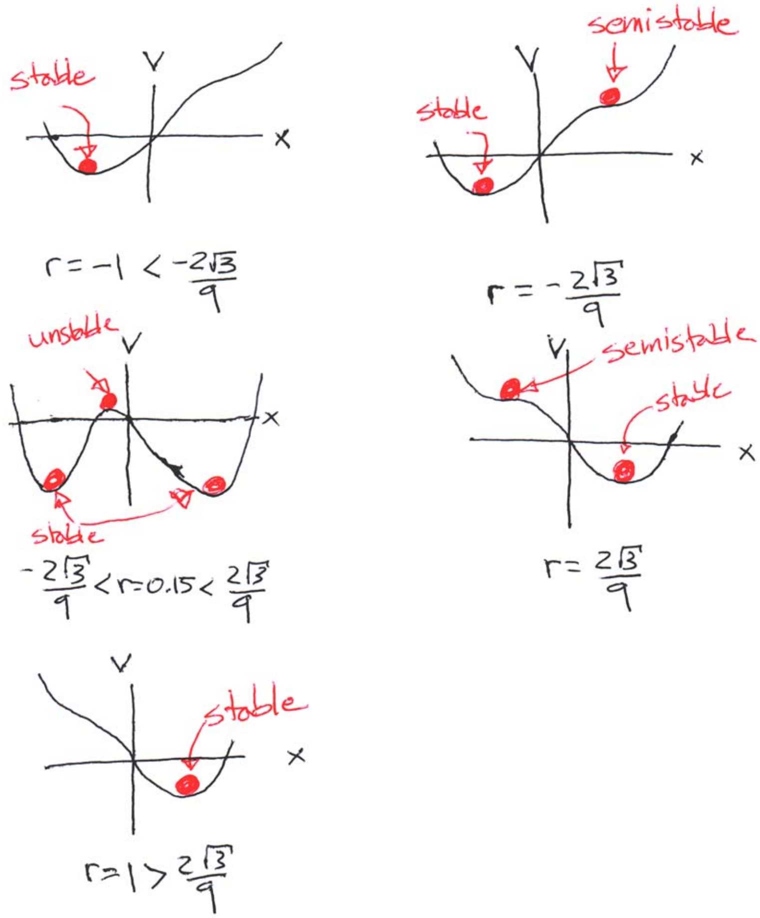
Before we start note that for $rx + x^3 > 1 + x^2$ it will suffice for $rx > 1$ and $x > 1$ (i.e., then it is obvious that the relationship holds). However, given r and our x_0 this might not hold initially. So we first make the following comparison: $\dot{x} = rx + x^3 > rx$, which the solution to $\dot{x} = rx$ with initial value $x(0) = x_0$ is $x = x_0 e^{rt}$. So initially our solution is growing at least as fast as an exponential function. In particular in finite time we know that $x_0 e^{rt}$ will be greater than both $1/r$ and 1.

What this means is that starting our solution we can let it grow and in a finite time we will have $x > 1/r$ and $x > 1$. At this point we then have that our rate of growth $\dot{x} > 1 + x^2$ and since we know $\dot{x} = 1 + x^2$ goes to ∞ in finite time and this is growing faster it follows that our solution must also go to ∞ in finite time.

2.7.1 Starting with $\dot{x} = x(1 - x) = x - x^2 = -dV/dx$ we see that a potential function is $V(x) = -(1/2)x^2 + (1/3)x^3$ (here we chose our constant of integration to be 0). This function has equilibrium points at $x = 0$ and $x = 1$. From the graph of the potential function we see that 0 is unstable and 1 is stable.



2.7.6 For $\dot{x} = r + x - x^3 = -dV/dx$ we see that the potential function is $V(x) = -rx - (1/2)x^2 + (1/4)x^4$. This potential function will have different behavior for different values of r. We will consider five different values of r and see the different behavior that can happen depending on how r relates to $\pm 2\sqrt{3}/9 \approx 0.3849$. These are shown in the pictures below.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V
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r = -1 < -2√3/9
semistable
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