Homework 2 solutions

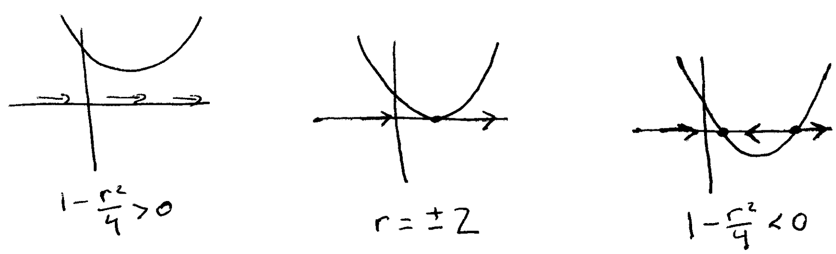
3.1.1 We start by examining $\dot{x} = 1 + rx + x^2$. The right hand side (as a function of x) is a parabola that is opening up. To get more information about the parabola we can rewrite this equation using completing the square to get

$$1 + rx + x^2 \;=\; x^2 + rx + \frac{r^2}{4} + 1 - \frac{r^2}{4} \;=\; \left(x + \frac{r}{2}\right)^2 + \left(1 - \frac{r^2}{4}\right).$$

In particular we now have three cases:

1) $1 - \frac{r^2}{4} > 0$ (or $|r| < 2$) in which case the parabola never hits the axis and there are no fixed points;
2) $1 - \frac{r^2}{4} = 0$ (or $r = \pm 2$) in which case the parabola hits the axis at a single (semistable fixed) point;
3) $1 - \frac{r^2}{4} < 0$ (or $|r| > 2$) in which case the parabola hits the axis at two points, the left corresponding to a stable and the right to an unstable fixed point.

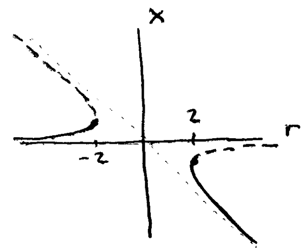
Pictures of the three cases are shown below.



We now see that there are 2 saddle node bifurcations. One happens at $r = -2$, $x = 1$ and the other at $r = 2$, $x = -1$. The branches are then the zeroes described by the quadratic equation $x^2 + rx + 1 = 0$ which are

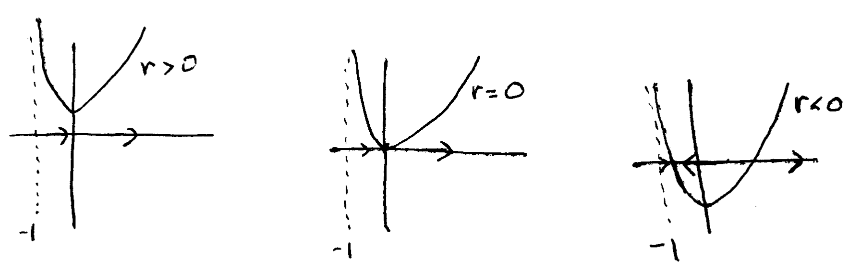
$$\frac{-r \pm \sqrt{r^2 - 4}}{2} \qquad \text{which for } r \text{ large is } \approx 0 \text{ or } -r.$$

Thus in the bifurcation diagram we have that $x = 0$ and $x = -r$ are asymptotes for the zeroes (in fact it turns out that the bifurcation diagram is a hyperbola). It follows from the discussion above that the bifurcation diagram has the following form (note from case (3) that the left, or lower value, is the stable one and the right, or higher value, is the unstable one; hence our marking).

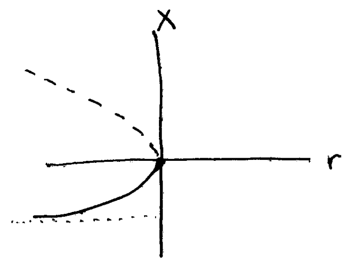


3.1.3 Looking at $\dot{x} = r + x - \ln(x+1)$ we see that the r acts as a *vertical shift* to the graph $x - \ln(x+1)$. So if we understand what the graph of $x - \ln(x+1)$ looks like then we can consider what happens as we shift the graph up and down, and then we will understand the behavior as we vary r. First off, $-\ln(x+1)$ will have a vertical asymptote at $x = -1$ and that the function will be coming down from $+\infty$, but then as x gets large the logarithm function will get small compared to x. What this means is that the graph of $x - \ln(x+1)$ comes down from $+\infty$ at $x = -1$ and then eventually starts to grow like x.

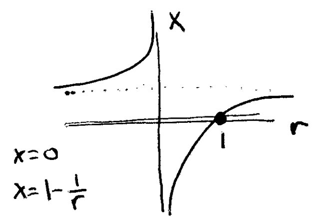
So the curve $x - \ln(x+1)$ comes down and then goes back up (much like a parabola). By taking the derivative and setting it $= 0$ we see that the minimum occurs at $x = 0$, and the function has minimal value 0. So the "bottom" of the curve is located at $(0, 0)$. Now if $r > 0$ then we shift the curve up and there are no intersections, while if $r < 0$ we shift the curve down and there are two intersections, the left (which will get close to -1) will correspond to a stable fixed point and the right (which for large r is approximately $-r$) will correspond to an unstable fixed point. Pictures of the three cases are shown below.



We now see that there is a saddle node bifurcation at $r = 0$, using the discussion from above we have that the bifurcation diagram will have the following form.



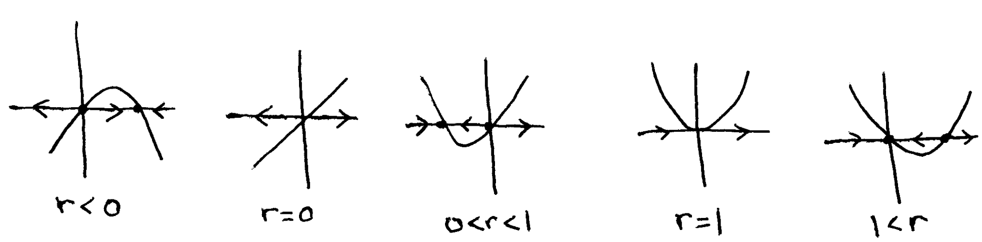
3.2.3 Starting with $\dot{x} = x - rx(1-x)$, or $(1-r)x + rx^2 = rx(x - \frac{r-1}{r})$, we can first sketch the graph underlying the bifurcation diagram to see what values of r to consider. In particular, we need to know when $x - rx(1-x) = x(1 - r(1-x)) = 0$. This will happen when $x = 0$ or $x = 1 - \frac{1}{r}$. From this we see that the graph underlying the bifurcation diagram is as follows:



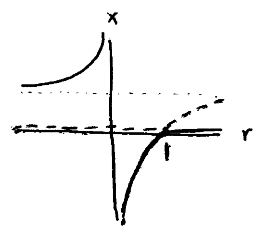
From this we see that the interesting r values are

1) $r < 0$, for which the parabola will open downwards with 0 as one unstable fixed point and a positive number as a second stable fixed point;
2) $r = 0$, for which it reduces to $\dot{x} = x$ which has only 0 as an unstable fixed point;
3) $0 < r < 1$, for which the parabola will open upwards with a negative stable fixed point and 0 as a second unstable fixed point;
4) $r = 1$, for which it reduces to $\dot{x} = x^2$ so that 0 is a semistable fixed point;
5) $1 < r$, for which the parabola will open upwards with 0 as a stable fixed point and a second positive unstable fixed point.

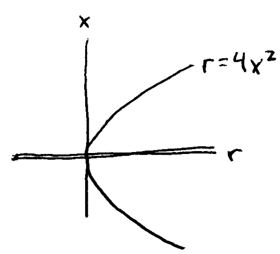
Pictures of the five cases are shown below.



With these it is now easy to mark the bifurcation diagram and so we have the following.



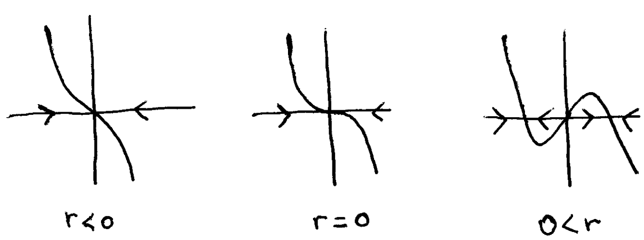
3.4.3 We have $\dot{x} = rx - 4x^3 = x(r - 4x^2)$. In order for this to be 0 we need to have $x = 0$ or $r = 4x^2$. This can help us to sketch the graph underlying the bifurcation diagram, which gives us the following.



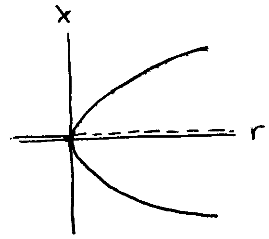
From this we see that the interesting r values are

1) $r < 0$, for which there is a single stable fixed point at 0;
2) $r = 0$, for which again there is a single stable fixed point at 0 (though the behavior in how quickly it goes to the stable fixed point is different, but this makes no difference though for the bifurcation diagram);
3) $r > 0$, for which there are three fixed points, a negative stable fixed point, a positive stable fixed point, and 0 which is now an unstable fixed point.

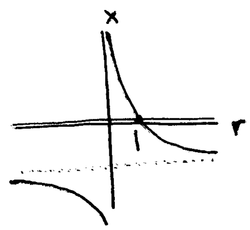
Pictures of the three cases are shown below.



With these it is now easy to mark the bifurcation diagram and so we have the following. In particular, we have a supercritical bifurcation at $r = 0$.



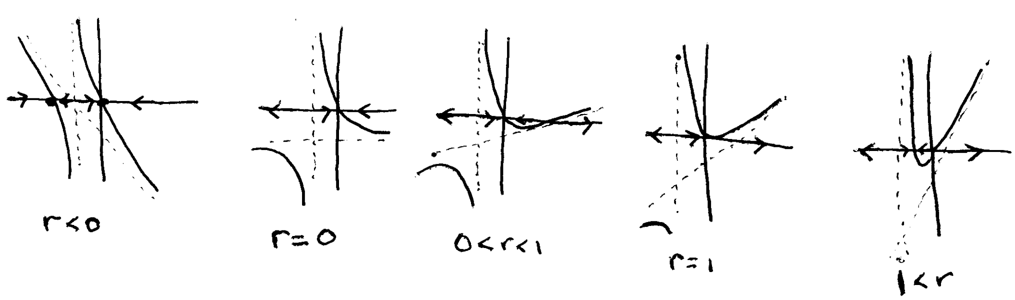
3.4.6 We have $\dot{x} = rx - x/(1+x) = x(r - 1/(1+x))$. In order for this to be 0 we need to have $x = 0$ or $r = 1/(1+x)$ which is equivalent to $x = (1/r) - 1$. This can help us to sketch the graph underlying the bifurcation diagram, which gives us the following. [Just by looking at the diagram we can anticipate that this will be a transcritical bifurcation.]



From this we see that the interesting r values are

1) $r < 0$, for which there are two stable fixed points, one negative and one 0;
2) $r = 0$, for which there is a single stable fixed point at 0;
3) $0 < r < 1$, for which there are two fixed points, one at 0 which is stable and a positive one which is unstable;
4) $r = 1$, for which there is a single semistable fixed point at 0;
5) $1 < r$, for which there are two fixed points, one negative which is stable and one at 0 which is unstable.

Pictures of the five cases are shown below.

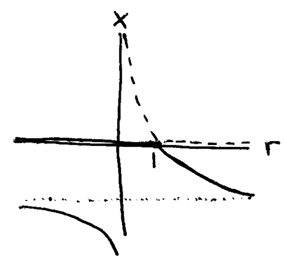


To help in drawing the above pictures note that

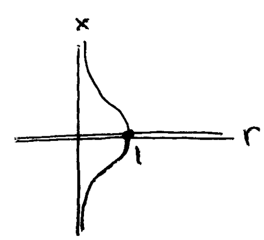
$$rx - \frac{x}{1+x} \;=\; rx - \frac{1+x-1}{1+x} \;=\; (rx-1) - \frac{1}{1+x}.$$

Thus in the pictures we have two asymptotes, a vertical asymptote occurring at $x = -1$ and a slant asymptote occuring at $rx - 1$ (i.e., for x "large" the $1/(1+x)$ term is small and not significant). So first the asymptotes were marked and then it was easy to get a sketch of the graph.

With these it is now easy to mark the bifurcation diagram and so we have the following. In particular, we see that as predicted this corresponds to a transcritical bifurcation at $r = 1$.



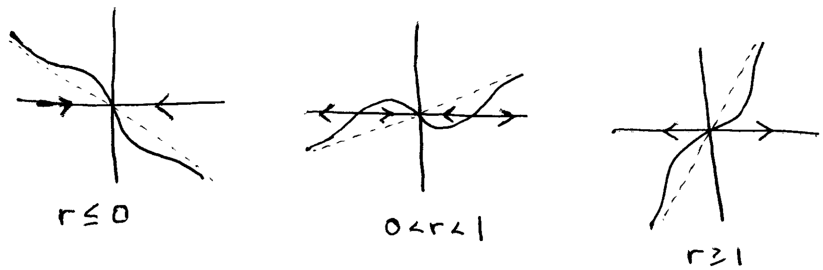
3.4.8 We have $\dot{x} = rx - x/(1+x^2) = x(r - 1/(1+x^2))$. In order for this to be 0 we need to have $x = 0$ or $r = 1/(1+x^2)$. This can help us to sketch the graph underlying the bifurcation diagram, which gives us the following. [Just by looking at the diagram we can anticipate that this will be some form of a pitchfork bifurcation.]



From this we see that the interesting r values are

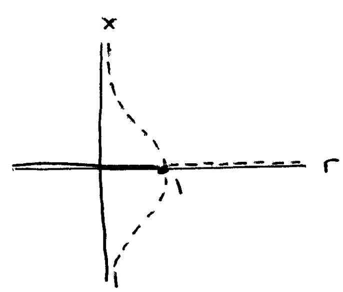
1) $r \le 0$, for which there is a single stable fixed point at 0;
2) $0 < r < 1$, for which there are three fixed points, a negative unstable root, a positive unstable root, and a stable root at 0;
3) $1 \le r$, for which there is a single unstable fixed point at 0.

Pictures of the three cases are shown below.



To help in drawing the above pictures note that we can think of this as adding the y values of the graph of $-x/(1+x^2)$ (which is a simple graph to draw) and rx.

With these it is now easy to mark the bifurcation diagram and so we have the following. In particular, we see that as predicted this corresponds to a subcritical pitchfork bifurcation at $r = 1$.



3.4.10 We can repeat the same procedure as in previous problems, but instead we will "tweak" it and see that we are already done. First note that

$$\dot{x} = rx + \frac{x^3}{1+x^2} = rx + \frac{x^3 + x - x}{1+x^2} = rx + x - \frac{x}{1+x^2} = \underbrace{(r+1)}_{=R} x - \frac{x}{1+x^2}.$$

So that this is the same as solving $\dot{x} = Rx - x/(1+x^2)$ and then shifting all the values of R by 1, but that was done in 3.4.8 and so there will be a subcritical pitchfork bifurcation at $r = 0$, similarly the bifurcation diagram is the same as in 3.4.8 shifted to the left by 1.

3.4.14 a) We have that

$$\dot{x} = rx + x^3 - x^5 = -x(x^4 - x^2 - r) = x\big((x^2)^2 - x^2 - r\big).$$

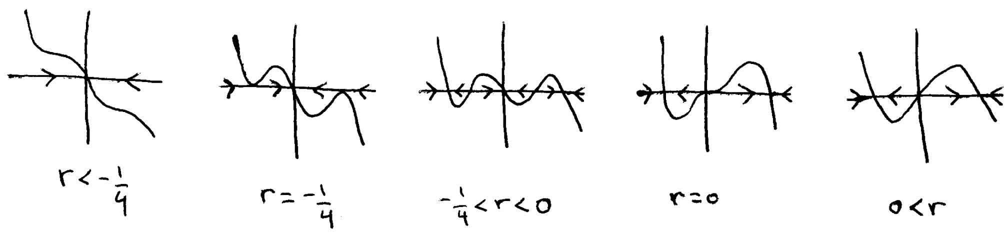
If we now apply the quadratic formula to the last part of the expression we have that

$$x^2 = \frac{1 \pm \sqrt{1+4r}}{2}.$$

It follows that the five roots are given by

$$0,\ \sqrt{\frac{1+\sqrt{1+4r}}{2}},\ \sqrt{\frac{1-\sqrt{1+4r}}{2}},\ -\sqrt{\frac{1+\sqrt{1+4r}}{2}},\ -\sqrt{\frac{1-\sqrt{1+4r}}{2}}.$$

b) Examining the roots in (a) we see that for $r < -1/4$ there will be one real root, at $r = -1/4$ there will be three real roots, for $-1/4 < r < 0$ there will be five real roots, and for $0 \le r$ there will be three real roots. This indicates the interesting places to sketch vector fields and we draw some below.



c) Looking at the answers in part (a) we see that when $1+4r < 0$ then all the nonzero roots will be imaginary, while when $1+4r = 0$ there will be two new nonzero roots (namely $\pm\sqrt{1/2}$). Therefore the parameter value at which the nonzero fixed points are born is $r_s = -1/4$.

3.5.2 We are considering $\dot{\phi} = \sin\phi(\gamma\cos\phi - 1) = f(\phi)$, where $\gamma > 0$ and ϕ between $-\pi$ and π. The zeroes of this equation will correspond to when either $\sin\phi = 0$, i.e., $\phi = -\pi, 0, \pi$, or $\cos\phi = 1/\gamma$. In the latter case if $\gamma < 1$ there will be no solution, while if $\gamma = 1$ then there is a single solution of $\phi = 0$, and finally if $\gamma > 1$ then there will be two solutions that get closer and closer to $\pm\pi/2$. Thus the zeroes are the same as in Figure 3.5.6. It now remains to check the stability.

Since $f'(\phi) = \cos\phi(\gamma\cos\phi - 1) - \gamma\sin^2\phi = \gamma\cos(2\phi) - \cos\phi$. We see that for all γ that $f'(\pi), f'(-\pi) = \gamma + 1 > 0$. This shows that $\phi = \pi$ and $\phi = -\pi$ are unstable. We also have that $f'(0) = \gamma - 1$ so that 0 will be stable for $\gamma < 1$ and unstable for $\gamma > 1$ (it is not difficult to show that for $\gamma = 1$ that $f(\phi) = -x^3/2 + O(x^5)$ so that for $\gamma = 1$ we have that the point $\phi = 0$ is also stable).

The only cases now left to consider is when $\gamma > 1$ and we have $\cos\phi_0 = 1/\gamma$. In this case note that $\cos(2\phi) = 2\cos^2\phi - 1 = (2/\gamma^2) - 1$. So we have at ϕ_0

$$f'(\phi_0) = \gamma\cos(2\phi_0) - \cos\phi_0 = \gamma\big(\frac{2}{\gamma^2} - 1\big) - \frac{1}{\gamma} = \frac{1}{\gamma} - \gamma < 0.$$

In particular these will correspond to stable fixed points.

Putting everything together we get the bifurcation diagram as shown in Figure 3.5.6.

3.5.3 We have that $\dot{\phi} = \sin\phi(\gamma\cos\phi - 1)$. From calculus it is known that for ϕ small that $\sin\phi = \phi - \phi^3/6 + O(\phi^5)$ while $\cos\phi = 1 - \phi^2/2 + O(\phi^4)$. If we substitute these in we then have (with a little bit of algebra)

$$\begin{aligned} \sin\phi(\gamma\cos\phi - 1) &= \big(\phi - \frac{1}{6}\phi^3 + O(\phi^5)\big)\big((\gamma-1) - \frac{\gamma}{2}\phi^2 + O(\phi^4)\big) \\ &= (\gamma-1)\phi - \big(\frac{\gamma}{2} + \frac{\gamma-1}{6}\big)\phi^3 + O(\phi^5) \\ &= \underbrace{(\gamma-1)}_{=A}\phi - \underbrace{\big(\frac{2}{3}\gamma - \frac{1}{6}\big)}_{=B}\phi^3 + O(\phi^5). \end{aligned}$$

3.5.6 a) We are considering the initial value problem $\varepsilon\ddot{x} + \dot{x} + x = 0$, with $x(0) = 1$ and $\dot{x}(0) = 0$. This is a constant coefficient linear equation, so we can apply some standard techniques. The characteristic equation then is $\varepsilon r^2 + r + 1$ which will have roots

$$\frac{-1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon}.$$

The way that we now solve this depends on what kind of roots we have. Namely there are three possibilities and we will go through each one.

1) $\varepsilon > 1/4$: In this case we see that we have two complex roots and so the general form of our solution will be

$$x(t) = e^{-t/2\varepsilon}\bigg(A\cos\big(\frac{\sqrt{4\varepsilon-1}}{2\varepsilon}t\big) + B\sin\big(\frac{\sqrt{4\varepsilon-1}}{2\varepsilon}t\big)\bigg).$$

Our initial condition $x(0) = 1$ then translates into $A = 1$. Taking the derivative we have that

$$\begin{aligned} \dot{x}(t) = \frac{-1}{2\varepsilon}e^{-t/2\varepsilon}&\bigg(\cos\big(\frac{\sqrt{4\varepsilon-1}}{2\varepsilon}t\big) + B\sin\big(\frac{\sqrt{4\varepsilon-1}}{2\varepsilon}t\big)\bigg) \\ &+ \frac{\sqrt{4\varepsilon-1}}{2\varepsilon}e^{-t/2\varepsilon}\bigg(-\sin\big(\frac{\sqrt{4\varepsilon-1}}{2\varepsilon}t\big) + B\cos\big(\frac{\sqrt{4\varepsilon-1}}{2\varepsilon}t\big)\bigg). \end{aligned}$$

Our initial condition $\dot{x}(0) = 0$ then translates into $(-1/2\varepsilon) + (\sqrt{4\varepsilon-1}/2\varepsilon)B = 0$ or $B = 1/\sqrt{4\varepsilon-1}$. So we have that the solution will be

$$x(t) = e^{-t/2\varepsilon}\bigg(\cos\big(\frac{\sqrt{4\varepsilon-1}}{2\varepsilon}t\big) + \frac{1}{\sqrt{4\varepsilon-1}}\sin\big(\frac{\sqrt{4\varepsilon-1}}{2\varepsilon}t\big)\bigg).$$

2) $\varepsilon = 1/4$: In this case we see that we have a repeated root of $r = -2$, and so the general form of our solution will be

$$x(t) = (At + B)e^{-2t}.$$

Our initial condition $x(0) = 1$ translates into $B = 1$. Since $\dot{x}(t) = (-2At + A - 2B)e^{-2t}$, our initial condition $\dot{x}(0) = 0$ translates into $A - 2B = 0$ so $A = 2B = 2$. So we have that the solution will be

$$x(t) = (2t + 1)e^{-2t}.$$

3) $0 < \varepsilon < 1/4$: In this case we see that we have two distinct real roots, and so the general form of our solution will be

$$x(t) = Ae^{(-1+\sqrt{1-4\varepsilon})t/2\varepsilon} + Be^{(-1-\sqrt{1-4\varepsilon})t/2\varepsilon}.$$

Our initial condition $x(0) = 1$ then translates into $A + B = 1$. Taking the derivative we have

$$\dot{x}(t) = \frac{-1 + \sqrt{1-4\varepsilon}}{2\varepsilon} Ae^{(-1+\sqrt{1-4\varepsilon})t/2\varepsilon} + \frac{-1 - \sqrt{1-4\varepsilon}}{2\varepsilon} Be^{(-1-\sqrt{1-4\varepsilon})t/2\varepsilon}.$$

Our initial condition $\dot{x}(0) = 0$ then translates into

$$\frac{-1 + \sqrt{1-4\varepsilon}}{2\varepsilon} A + \frac{-1 - \sqrt{1-4\varepsilon}}{2\varepsilon} B = 0$$

If we multiply both sides by 2ε, use the fact that $A + B = 1$ (given above) and move the 1 to the other side and then divide by $\sqrt{1-4\varepsilon}$ we are left with

$$A - B = \frac{1}{\sqrt{1-4\varepsilon}}.$$

It is now easy to solve for A and B to get

$$A = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1-4\varepsilon}}\right), \qquad B = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1-4\varepsilon}}\right),$$

so that the solution will be

$$x(t) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1-4\varepsilon}}\right)e^{(-1+\sqrt{1-4\varepsilon})t/2\varepsilon} + \frac{1}{2}\left(1 - \frac{1}{\sqrt{1-4\varepsilon}}\right)e^{(-1-\sqrt{1-4\varepsilon})t/2\varepsilon}.$$

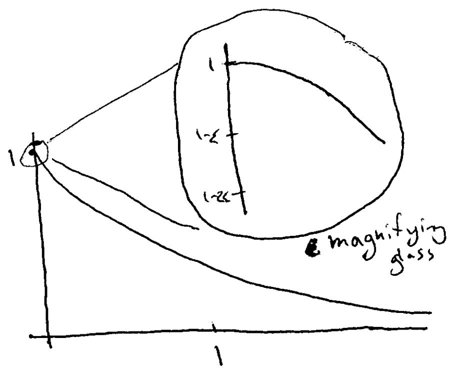
b) If $\varepsilon \ll 1$ (in other words ε is a very small number). We now look at the solutions in part (a). We can simplify the solution by noting that for ε very small that $\sqrt{1-4\varepsilon} \approx 1 - 2\varepsilon$ and $1/\sqrt{1-4\varepsilon} \approx 1 + 2\varepsilon$, so that our solution is approximately (for t small)

$$x(t) \approx e^{-t} - \varepsilon e^{-t/\varepsilon}.$$

In particular we see that this is a combination of two exponential functions, one with decay rate 1 and the other with decay rate $1/\varepsilon$. These decay rates give the two time scales involved, namely 1 and ϵ.

Another way to think of it is that the first term (e^{-t}) decays very slowly compared to the second term ($e^{-t/\varepsilon}$) in that the latter quickly decays to zero in time proportional to ε while the first decays to zero in time proportional to 1.

c) When the graph is very close to 0 (say within a few ϵ) it starts out flat and starts decreasing while the graph is also concave down. But soon the $e^{-t/\varepsilon}$ term decays out and the graph starts looking like e^{-t}. These are shown in the pictures below where we have looked at two time scales.



d) As we let ϵ go to zero the solution is approaching $x(t) = e^{-t}$ for $t > 0$ which is the solution to the initial value problem $\dot{x} + x = 0$ with $x(0) = 1$, so in that regard it works well. (Of course we note that e^{-t} fails miserably to the condition that $\dot{x}(0) = 0$, this condition seems to be what the $-\varepsilon e^{-t/\varepsilon}$ term is doing.) On the other hand, if we look at $t < 0$ we see that the solution blows up. So there are some problems for a general solution.