Homework 3 solutions

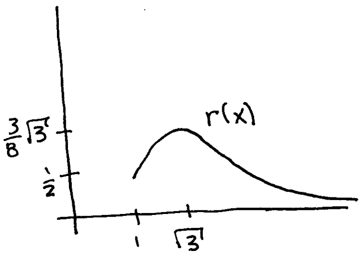
3.7.2 a) The graphs of

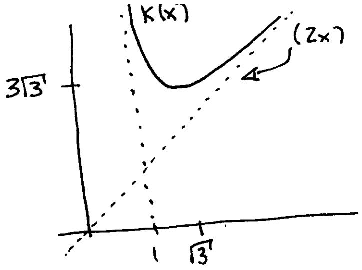
$$r(x) = \frac{2x^3}{(1+x^2)^2} \qquad \text{and} \qquad k(x) = \frac{2x^3}{x^2-1}$$

are shown below. Note that since the derivatives of these functions are

$$r'(x) = \frac{2x^2(3-x^2)}{(1+x^2)^3} \qquad \text{and} \qquad k'(x) = \frac{2x^2(x^2-3)}{(x^2-1)^2}$$

respectively, it is simple to see that for $x > 1$ they both have only one critical point. For both functions it occurs at $x = \sqrt{3}$ and for $r(x)$ it is a local max and for $k(x)$ it is a local min (i.e., by using the first derivative test for instance). This explains some of the structure that is indicated.





b) The cusp occurs where both functions "reverse" direction which corresponds to the critical points. As noted above both of these functions have a critical point at $x = \sqrt{3}$, plugging this into the function we have $k = 3\sqrt{3} \approx 5.1965$ and $r = \frac{3}{8}\sqrt{3} \approx 0.6495$ at the cusp.

3.7.3 a) Starting with

$$\dot{N} = \frac{dN}{dt} = rN\big(1 - \frac{N}{K}\big) - H,$$

substitute $x = N/K$, so that $dN/dt = K\,dx/dt$ so the above becomes

$$K\frac{dx}{dt} = rKx(1-x) - H \qquad \text{or} \qquad \frac{1}{r}\frac{dx}{dt} = x(1-x) - \underbrace{\frac{H}{rK}}_{=h}.$$

Finally if we let $\tau = rt$ so that $d\tau = r\,dt$ then this becomes

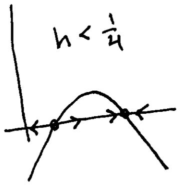
$$\frac{dx}{d\tau} = x(1-x) - h,$$

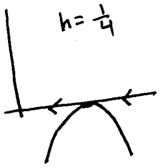
as desired.

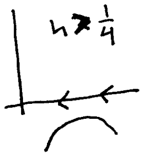
b) Note that h now corresponds to a vertical shift of the graph up or down. Since the graph of $x(1-x)$ is a downward opening parabola with a maximum height of $1/4$ at $x = 1/2$ we see that the interesting ranges for h will be:

1) $h < 1/4$: for which there are two fixed points, the left one unstable and the right one stable;
2) $h = 1/4$: for which there is a single fixed point at $x = 1/2$ which is semistable;
3) $h > 1/4$: for which there are no fixed points and everything moves left.

Examples of these three ranges are shown below.







c) From the discussion above we see that there will be a saddle node bifurcation at $h_c = 1/4$.

d) In the case that $h_c < 1/4$ there are two possibilities. If the population is initially small (i.e., $x < (1-\sqrt{1-4h})/2$) then the harvesting will drive the fish population to 0, otherwise the fish population will move to a steady equilibrium (i.e., $x = (1+\sqrt{1-4h})/2$).

When $h_c > 1/4$ then the fish population will experience a steady decline and eventually will be driven to 0.

3.7.4 a) If we look at $N/(A+N)$ the smaller that A is compared to N the closer this is to 1 and the larger that A is compared to N the closer this is to 0. So if we think of H as the harvest goal then the term $N/(A+N)$ gives some measurement of what proportion of the potential harvest is collected.

b) Starting with

$$\dot{N} = \frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - H\frac{N}{A+N},$$

we again substitute $x = N/K$, so that $dN/dt = K\,dx/dt$ so the above becomes

$$K\frac{dx}{dt} = rKx(1-x) - H\frac{Kx}{A+Kx} \qquad \text{or} \qquad \frac{1}{r}\frac{dx}{dt} = x(1-x) - \underbrace{\frac{H}{rK}}_{=h}\,\frac{x}{\underbrace{(A/K)}_{=a}+x}.$$

Finally if we let $\tau = rt$ so that $d\tau = r\,dt$ then this becomes

$$\frac{dx}{d\tau} = x(1-x) - h\frac{x}{a+x},$$

as desired.

c) First note that

$$\frac{dx}{dt} = x(1-x) - h\frac{x}{a+x} = x\big(1 - x - \frac{h}{a+x}\big) = 0$$

if either $x = 0$ or $1 - x - h/(a+x) = 0$. In the latter case this is equivalent to $(1-x)(a+x) - h = 0$ or $-x^2 - (a-1)x - (h-a) = 0$ which will give the other zeroes of

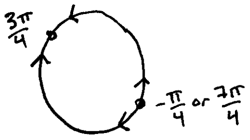
$$\frac{(1-a) \pm \sqrt{(a-1)^2 - 4(h-a)}}{2}.$$

Thus we have three possibilities depending on whether the term in the square root is negative, zero or positive, which for fixed points will give either one (a stable fixed point at 0), two (which can be a positive or negative semistable point and a stable fixed point at 0), or three (a negative stable fixed point, an unstable fixed point at 0 and a positive stable fixed point).

4.1.5 We have that $\dot{\theta} = \sin\theta + \cos\theta$. By using some trigonometry this can be rewritten as

$$\dot{\theta} = \sqrt{2}\big(\frac{\sqrt{2}}{2}\sin\theta + \frac{\sqrt{2}}{2}\cos\theta\big) = \sqrt{2}\big(\cos\frac{\pi}{4}\sin\theta + \sin\frac{\pi}{4}\cos\theta\big) = \sqrt{2}\sin\big(\theta + \frac{\pi}{4}\big).$$

Thus on the circle we have for $-\pi/4 < \theta < 3\pi/4$ that $\dot{\theta} > 0$ while for $3\pi/4 < \theta < 7\pi/4$ we have $\dot{\theta} < 0$. From this it follows that $3\pi/4$ is stable and $-\pi/4$ (or $7\pi/4$) is unstable. The vector field on the circle is shown below.



4.3.2 a) If we let $u = \tan(\theta/2)$ then we have that $\theta/2 = \arctan u$ or $\theta = 2\arctan u$. Taking derivatives we then have

$$d\theta = \frac{2du}{1+u^2}.$$

b) Drawing a triangle with angle $\theta/2$ and legs of length 1 and u (as suggested in the hint), we see that the hypotenuse of the triangle will be $\sqrt{1+u^2}$ so that

$$\sin\frac{\theta}{2} = \frac{u}{\sqrt{1+u^2}} \qquad \text{and} \qquad \cos\frac{\theta}{2} = \frac{1}{\sqrt{1+u^2}}.$$

From this it follows that

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\big(\frac{u}{\sqrt{1+u^2}}\big)\big(\frac{1}{\sqrt{1+u^2}}\big) = \frac{2u}{1+u^2}.$$

c) Since the tangent function has asymptotes at $\pm\pi/2$ we see that as $\theta \to \pi$ that $u \to \infty$ and similarly as $\theta \to -\pi$ then $u \to -\infty$.

d) Using the information about u from above we now have

$$\begin{aligned} T &= \int_{\theta=-\pi}^{\theta=\pi} \frac{d\theta}{\omega - a\sin\theta} = \int_{u=-\infty}^{u=\infty} \frac{2du/(1+u^2)}{\omega - a(2u/(1+u^2))} \\ &= \frac{2}{\omega}\int_{-\infty}^{\infty} \frac{du}{u^2 - (2a/\omega)u + 1}. \end{aligned}$$

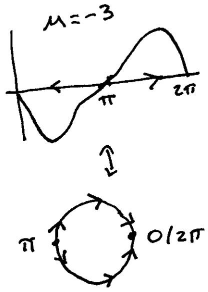
e) Continuing the work in the previous section we have

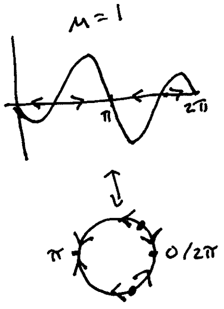
$$\begin{aligned} T &= \frac{2}{\omega}\int_{-\infty}^{\infty} \frac{du}{u^2 - (2a/\omega)u + (a^2/\omega^2) + 1 - (a^2/\omega^2)} \\ &= \frac{2}{\omega}\int_{-\infty}^{\infty} \frac{du}{\underbrace{(u - a/\omega)}_{=x}{}^2 + \underbrace{1 - (a^2/\omega^2)}_{=r}} = \frac{2}{\omega}\int_{-\infty}^{\infty} \frac{dx}{x^2 + r} \\ &= \frac{2}{\omega}\frac{\pi}{\sqrt{1 - (a^2/\omega^2)}} = \frac{2\pi}{\sqrt{\omega^2 - a^2}}. \end{aligned}$$

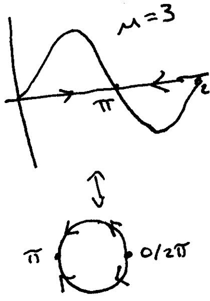
4.3.3 We consider

$$\dot{\theta} = \mu\sin\theta - \sin(2\theta) = \mu\sin\theta - 2\sin\theta\cos\theta = \sin\theta\big(\mu - 2\cos\theta\big).$$

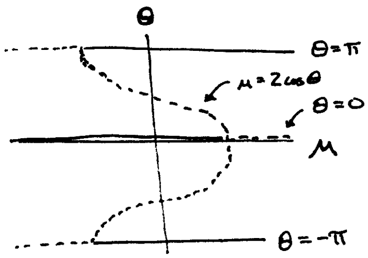
In particular we see that the fixed points correspond to either $\sin\theta = 0$, i.e., $\theta = -\pi, 0, \pi$ or $\mu = 2\cos\theta$, i.e., $\theta = \arccos(\mu/2)$. In particular we see that for $|\mu| \geq 2$ there will be two fixed points at 0 and $\pm\pi$ while for $|\mu| < 2$ there will be four fixed points. Examples of the different possibilities are shown below.



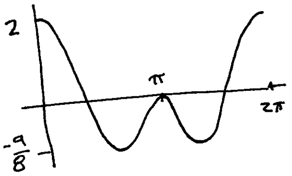




From this it is easy to fill in the phase portrait. In particular we see that there is a substable pitchfork bifurcation at $\mu = -2$ and $\theta = \pm\pi$ and a substable pitchfork bifurcation at $\mu = 2$ and $\theta = 0$.



4.3.5 We consider $\dot{\theta} = \mu + \cos\theta + \cos(2\theta)$. In this case the μ parameter can be thought of as acting as a "vertical shift" to the graph of $\cos\theta + \cos(2\theta)$. By using some calculus (i.e., looking at the zeroes of the derivative $-\sin\theta - 4\sin\theta\cos\theta$) we can see that for θ in the range $[0, 2\pi)$ it starts out at $\theta = 0$ with a maximum of 2, decreases to $\theta = \arccos(-1/4) \approx 1.8234$ to a minimum of $-9/8$, increases again to $\theta = \pi$ to a maximum of 0, decreases to $\theta = 2\pi - \arccos(-1/4) \approx 4.4597$ to a minimum of $-9/8$ and finally increases again to $\theta = 2\pi$ with a maximum of 2. A graph of $\cos\theta + \cos(2\theta)$ is shown below.



From this we see the following interesting ranges:

a) $\mu < -2$: there are no fixed points and $\dot{\theta} < 0$ for all θ;

b) $\mu = -2$: there is a single semistable fixed point at $\theta = 0$;

c) $-2 < \mu < 0$: there are two fixed points, one stable and one unstable;

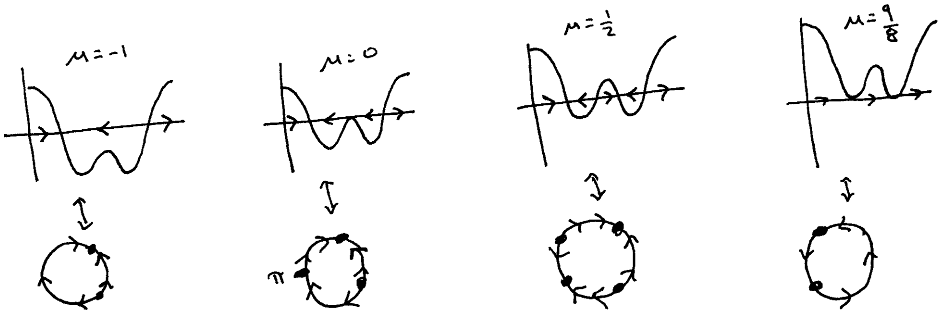
d) $\mu = 0$: there are three fixed points, one stable, one unstable and one semistable;

e) $0 < \mu < 9/8$: there are four fixed points which alternate between stable and unstable;

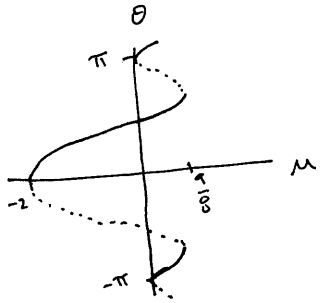
f) $\mu = 9/8$: there are two fixed points, both of which are semistable;

g) $\mu > 9/8$: there are no fixed points and $\dot{\theta} > 0$ for all θ.

Phase portraits of some of these cases are shown below.



From this it is easy to draw the phase diagram to get the following.



In particular we see that there are four saddle node bifurcations. One at $\mu = -2$ and $\theta = 0$; one at $\mu = 0$ and $\theta = \pi$ and two at $\mu = 9/8$ and $\theta = \arccos(-1/4), 2\pi - \arccos(-1/4)$.