Homework 4 solutions

5.2.1 a) Starting with $\dot{x} = 4x - y$ and $\dot{y} = 2x + y$ we have

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 4x - y \\ 2x + y \end{bmatrix} = \underbrace{\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}}_{=A} \begin{bmatrix} x \\ y \end{bmatrix} = A\mathbf{x}.$$

The characteristic polynomial is then given by

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{bmatrix}\right) = (4-\lambda)(1-\lambda) - (2)(-1) = \lambda^2 - 5\lambda + 6.$$

The eigenvalues are the roots of the characteristic polynomial and since $\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$ the eigenvalues are 2 and 3. To find the corresponding eigenvectors we can use the results of the homework problem below (or other methods) to show that an eigenvector associated with eigenvalue 2 is $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$ and an eigenvector associated with eigenvalue 3 is $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$.

b) From the answer in the previous part it is easy to give the general solution

$$\mathbf{x} = Ce^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + De^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

c) Since both eigenvalues are positive then we have an unstable node at the origin.

d) Plugging in the initial condition we have that

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \mathbf{x}_0 = C\begin{bmatrix} 1 \\ 2 \end{bmatrix} + D\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} C + D \\ 2C + D \end{bmatrix}.$$

So we need that $C + D = 3$ and $2C + D = 4$. Subtracting the two we have that $C = 1$ and then we need $D = 2$. So we have that the solution is

$$\mathbf{x} = e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{2t} + 2e^{3t} \\ 2e^{2t} + 2e^{3t} \end{bmatrix}.$$

5.2.2 a) The system $\dot{x} = x - y$ and $\dot{y} = x + y$ can be rewritten as

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\mathbf{x},$$

we then have that the characteristic polynomial (used to find the eigenvalues) is

$$\det\left(\begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix}\right) = \lambda^2 - 2\lambda + 2 \qquad \text{roots} = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i.$$

To find the corresponding eigenvectors we can use the results of the homework problem below (or other methods) to show that an eigenvector associated with eigenvalue $1+i$ is $\left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]$ and an eigenvector associated with eigenvalue $1-i$ is $\left[\begin{smallmatrix} -i \\ 1 \end{smallmatrix}\right]$.

b) Using that $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ we have that the solution is

$$\begin{aligned}
\mathbf{x} &= ce^{(1+i)t}\begin{bmatrix} i \\ 1 \end{bmatrix} + de^{(1-i)t}\begin{bmatrix} -i \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} ce^te^{it}i - de^te^{-it}i \\ ce^te^{it} + de^te^{-it} \end{bmatrix} \\
&= \begin{bmatrix} ce^t\big(\cos(t) + i\sin(t)\big)i - de^t\big(\cos(t) - i\sin(t)\big)i \\ ce^t\big(\cos(t) + i\sin(t)\big) + de^t\big(\cos(t) - i\sin(t)\big) \end{bmatrix} \\
&= \begin{bmatrix} (-c-d)e^t\sin(t) + (c-d)e^t\cos(t)i \\ (c+d)e^t\cos(t) + (c-d)e^t\sin(t)i \end{bmatrix} \\
&= \underbrace{(c+d)}_{=C}e^t\begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} + \underbrace{(c-d)i}_{=D}e^t\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \\
&= Ce^t\begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} + De^t\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}.
\end{aligned}$$

5.2.4 We have $\dot{x} = 5x + 10y$ and $\dot{y} = -x - y$ which in matrix form is

$$\dot{\mathbf{x}} = \begin{bmatrix} 5 & 10 \\ -1 & -1 \end{bmatrix}\mathbf{x}.$$

The characteristic polynomial of this matrix is $\lambda^2 - 4\lambda + 5$ so that the eigenvalues are $2 \pm i$. Using the homework result below we have that an eigenvector associated with the eigenvalue $2+i$ is $\left[\begin{smallmatrix} 3+i \\ -1 \end{smallmatrix}\right]$ while an eigenvector associated with the eigenvalue $2-i$ is $\left[\begin{smallmatrix} 3-i \\ -1 \end{smallmatrix}\right]$. So we have that the solution is

$$\begin{aligned}
\mathbf{x} &= ce^{(2+i)t}\begin{bmatrix} 3+i \\ -1 \end{bmatrix} + de^{(2-i)t}\begin{bmatrix} 3-i \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} ce^{2t}\big(\cos(t) + i\sin(t)\big)(3+i) \\ -ce^{2t}\big(\cos(t) + i\sin(t)\big) \end{bmatrix} + \begin{bmatrix} de^{2t}\big(\cos(t) - i\sin(t)\big)(3-i) \\ -de^{2t}\big(\cos(t) - i\sin(t)\big) \end{bmatrix} \\
&= \begin{bmatrix} (c+d)(3\cos(t) - \sin(t))e^{2t} + i(c-d)(3\sin(t) + \cos(t))e^{2t} \\ (c+d)(-\cos(t))e^{2t} + i(c-d)(-\sin(t))e^{2t} \end{bmatrix} \\
&= \underbrace{(c+d)}_{=C}e^{2t}\begin{bmatrix} 3\cos(t) - \sin(t) \\ -\cos(t) \end{bmatrix} + \underbrace{i(c-d)}_{=D}e^{2t}\begin{bmatrix} 3\sin(t) + \cos(t) \\ -\sin(t) \end{bmatrix} \\
&= Ce^{2t}\begin{bmatrix} 3\cos(t) - \sin(t) \\ -\cos(t) \end{bmatrix} + De^{2t}\begin{bmatrix} 3\sin(t) + \cos(t) \\ -\sin(t) \end{bmatrix}.
\end{aligned}$$

5.2.5 We have $\dot{x} = 3x - 4y$ and $\dot{y} = x - y$ which in matrix form is

$$\dot{\mathbf{x}} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\mathbf{x}.$$

The characteristic polynomial of this matrix is $\lambda^2 - 2\lambda + 1$ so that the eigenvalues are $1, 1$ (i.e., a repeated eigenvalue). It is easy to check that the only eigenvector (up to scaling) associated with the eigenvalue 1 is $\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$. We have a deficiency of eigenvectors so we will need to do some work to get the solution. First note that we already know one solution is $\mathbf{x} = e^t \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$. In analogy with repeated roots for a linear differential equation a guess for the second solution might be $\mathbf{x} = te^t \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$. If we put this guess in we would have

$$\underbrace{te^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{=\dot{x}} \stackrel{?}{=} \underbrace{te^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{=A\mathbf{x}}.$$

This clearly doesn't work but perhaps we are not too far off, the problem is that we have an extra term on the left hand side. So now let us modify our guess to be $\mathbf{x} = te^t \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right] + e^t\mathbf{y}$ where we will choose $\mathbf{y}$ so that the relationship is satisfied. In this case we have that

$$\underbrace{te^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^t \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \mathbf{y} \right)}_{=\dot{x}} \stackrel{?}{=} \underbrace{te^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^t A\mathbf{y}}_{=A\mathbf{x}}.$$

In particular if we choose $\mathbf{y}$ so that $A\mathbf{y} - \mathbf{y} = \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$ then the above relationship will be satisfied. Let $\mathbf{y} = \left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$ then we need to have

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

which reduces to needing $a = 2b + 1$ so that the above relationship is satisfied for

$$\mathbf{y} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2b + 1 \\ b \end{bmatrix} = b \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The first part of $\mathbf{y}$ (the $b \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$) will correspond to the first solution that we found and so we can drop it. Putting it all together we then have that our general solution is

$$\mathbf{x} = Ce^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + D\left(te^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = Ce^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + De^t \begin{bmatrix} 2t + 1 \\ t \end{bmatrix}.$$

5.2.6 We have $\dot{x} = -3x + 2y$ and $\dot{y} = x - 2y$ which in matrix form is

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}.$$

The characteristic polynomial of this matrix is $\lambda^2 + 5\lambda + 4$ so that the eigenvalues are $-4, -1$. Using the homework result below we have that an eigenvector associated with the eigenvalue -4 is $\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right]$ while an eigenvector associated with the eigenvalue -1 is $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$. So we have that the general solution is

$$\mathbf{x} = Ce^{-4t}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + De^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

5.2.7 We have $\dot{x} = 5x + 2y$ and $\dot{y} = -17x - 5y$ which in matrix form is

$$\dot{\mathbf{x}} = \begin{bmatrix} 5 & 2 \\ -17 & -5 \end{bmatrix}\mathbf{x}.$$

The characteristic polynomial of this matrix is $\lambda^2 + 9$ so that the eigenvalues are $\pm 3i$. Using the homework result below we have that an eigenvector associated with the eigenvalue $-3i$ is $\left[\begin{smallmatrix}5-3i\\-17\end{smallmatrix}\right]$ and an eigenvector associated with the eigenvalue $3i$ is $\left[\begin{smallmatrix}5+3i\\-17\end{smallmatrix}\right]$. So we have that the solution is

$$\begin{aligned}
\mathbf{x} &= ce^{-3ti}\begin{bmatrix} 5-3i \\ -17 \end{bmatrix} + de^{3ti}\begin{bmatrix} 5+3i \\ -17 \end{bmatrix} \\
&= \begin{bmatrix} c\big(\cos(3t) - i\sin(3t)\big)(5-3i) \\ -17c\big(\cos(3t) - i\sin(3t)\big) \end{bmatrix} + \begin{bmatrix} d\big(\cos(3t) + i\sin(3t)\big)(5+3i) \\ -17d\big(\cos(3t) + i\sin(3t)\big) \end{bmatrix} \\
&= \begin{bmatrix} (c+d)(5\cos(3t) - 3\sin(3t)) + (c-d)i(-3\cos(3t) - 5\sin(3t)) \\ (c+d)(-17\cos(3t)) + (c-d)i(17\sin(3t)) \end{bmatrix} \\
&= \underbrace{(c+d)}_{=C}\begin{bmatrix} 5\cos(3t) - 3\sin(3t) \\ -17\cos(3t) \end{bmatrix} + \underbrace{(c-d)i}_{=D}\begin{bmatrix} -3\cos(3t) - 5\sin(3t) \\ 17\sin(3t) \end{bmatrix} \\
&= C\begin{bmatrix} 5\cos(3t) - 3\sin(3t) \\ -17\cos(3t) \end{bmatrix} + D\begin{bmatrix} -3\cos(3t) - 5\sin(3t) \\ 17\sin(3t) \end{bmatrix}.
\end{aligned}$$

5.2.8 We have $\dot{x} = -3x + 4y$ and $\dot{y} = -2x + 3y$ which in matrix form is

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix}\mathbf{x}.$$

The characteristic polynomial of this matrix is $\lambda^2 - 1$ so that the eigenvalues are ± 1. Using the homework result below we have that an eigenvector associated with the eigenvalue -1 is $\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$ while an eigenvector associated with the eigenvalue 1 is $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$. So we have that the general solution is

$$\mathbf{x} = Ce^{-t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + De^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

* Let λ and μ be the eigenvalues of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We will make use of the following facts about eigenvalues (discussed in section last week): $\lambda + \mu = a + d$ and $\lambda\mu = ad - bc$. Since a vector $\mathbf{x}$ is an eigenvector of A associated with eigenvalue μ if $A\mathbf{x} = \mu\mathbf{x}$, it suffices to show that this last relationship holds for the columns of $A - \lambda I$, then if the columns are nonzero they are eigenvectors. So consider the following,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a - \lambda \\ c \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \mu - d \\ c \end{bmatrix} = \begin{bmatrix} \mu a - ad + bc \\ \mu c - dc + dc \end{bmatrix} = \begin{bmatrix} \mu a - \lambda\mu \\ \mu c \end{bmatrix}$$
$$= \mu \begin{bmatrix} a - \lambda \\ c \end{bmatrix}.$$

Similarly we have the following,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} b \\ d - \lambda \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} b \\ \mu - a \end{bmatrix} = \begin{bmatrix} ab + \mu b - ab \\ bc + \mu d - ad \end{bmatrix} = \begin{bmatrix} \mu b \\ \mu d - \mu\lambda \end{bmatrix}$$
$$= \mu \begin{bmatrix} b \\ d - \lambda \end{bmatrix}.$$

** Set $x = aX$ and $t = c\tau$ (i.e., scaling the units of x and t). Then we have that

$$\frac{dx}{dt} = \frac{d(aX)}{d\tau}\frac{d\tau}{dt} = \frac{a}{c}\frac{dX}{d\tau} \quad \text{and} \quad \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{a}{c}\frac{d}{dt}\left(\frac{dX}{d\tau}\right) = \frac{a}{c}\frac{d^2X}{d\tau^2}\frac{d\tau}{dt} = \frac{a}{c^2}\frac{d^2X}{d\tau^2}.$$

Substituting, we then have that

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \qquad \text{becomes} \qquad \frac{ma}{c^2}\frac{d^2X}{d\tau^2} + \frac{ab}{c}\frac{dX}{d\tau} + akX.$$

In order for this last equation to become $\ddot{x} + B\dot{x} + 2x = 0$ we need to have $\frac{ma}{c^2} = 1$ and $ak = 2$. This is possible if we let $a = 2/k$ and $c = \sqrt{2m/k}$. In this case we have that

$$B = \frac{ab}{c} = \frac{\frac{2}{k}b}{\sqrt{\frac{2m}{k}}} = b\sqrt{\frac{2}{mk}}.$$