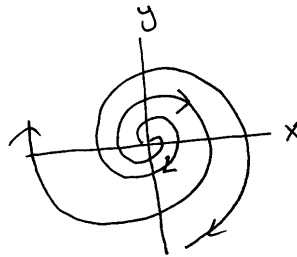
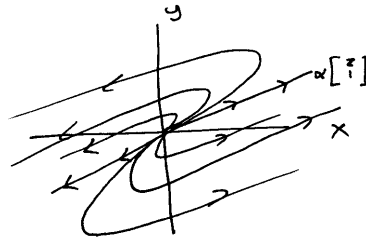


# Homework 5 solutions

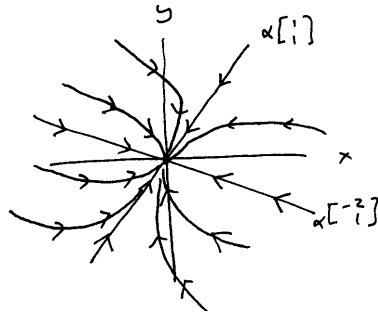
5.2.4 Since the trace is 4 and the determinant is 5, then the fixed point is an unstable spiral (alternatively this follows because the eigenvalues are  $2 + i$  and  $2 - i$ , since they are complex it spirals and since the real part is  $> 0$  it is unstable). The phase portrait are spirals which head out of the origin spinning *clockwise* (to see that it is clockwise we can, for example, plot a few points on a single trajectory to get the shape, alternatively we can see that along the positive  $x$ -axis that  $\dot{y} < 0$  which tells us how we cross the positive  $x$ -axis).



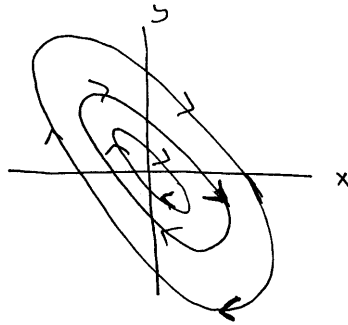
5.2.5 Since the trace is 2 and the determinant is 1 then the fixed point is a degenerate node (not a star because while we have repeated eigenvalues of 1, 1 we do not have a full set of eigenvectors). The only eigenvector is  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and is indicated below (since our eigenvalue is  $> 0$  then it moves out). To get the behavior of the remaining part we can plot a few points along a solution (i.e., such as  $e^t \begin{bmatrix} 2t+1 \\ t \end{bmatrix}$ ). The phase portrait is shown below.



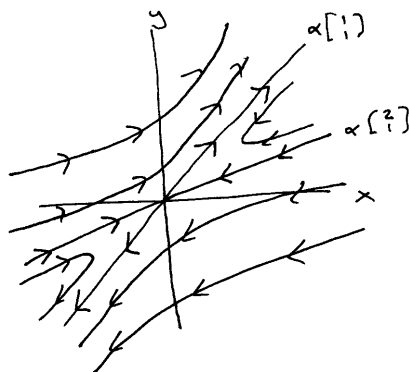
5.2.6 Since the trace is  $-5$  and the determinant is  $4$  then the fixed point is a stable node (alternatively since both eigenvalues are negative then solutions tend to the origin as  $t \rightarrow \infty$ ). The “fast” eigenvector is  $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$  while the “slow” eigenvector is  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Using this we have the following phase portrait.



5.2.7 Since the trace is  $0$  and the determinant is  $9$  then the fixed point corresponds to a neutrally stable center (alternatively since the eigenvalues are  $3i$  and  $-3i$ , since the real part is  $0$  there will be no exponential decay or growth and the solutions will be periodic). The phase portrait will consist of ellipses centered around the origin spinning *clockwise* (again to see that it spins clockwise we can plot a trajectory or again look at the positive  $x$ -axis and see that  $\dot{y} < 0$  which tells us how we cross that axis which will give us the direction).



5.2.8 Since the trace is 0 and the determinant is  $-1$  then the fixed point is a saddle (alternatively this follows since the eigenvalues are 1 and  $-1$ , i.e., real and differ in sign so that in one direction we move towards the origin and in another direction we move away from the origin). The eigenvector associated with 1 is  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and the eigenvector associated with  $-1$  is  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ . This gives us the following phase portrait.



5.2.11 The matrix

$$A = \begin{bmatrix} \lambda & b \\ 0 & \lambda \end{bmatrix}$$

has characteristic polynomial  $(x - \lambda)^2$  so that  $\lambda$  is the only eigenvalue of  $A$  and occurs as a repeated root. Looking for the eigenvectors we see that if  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  is an eigenvector then  $(A - \lambda I)\mathbf{x} = 0$ , i.e.,

$$\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

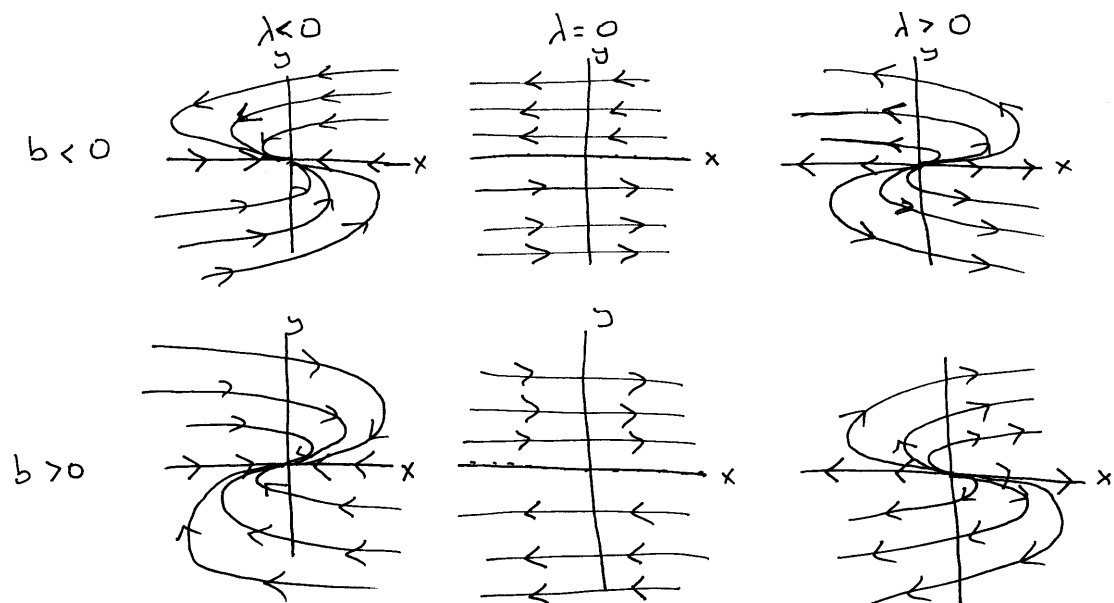
which reduces to the condition that  $b x_2 = 0$  and since  $b \neq 0$  we must conclude that  $x_2 = 0$ . So we only have a one dimensional eigenspace and it consists of all vectors of the form  $\mathbf{x} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ .

To solve the system we have the obvious solution  $\mathbf{x} = ce^{\lambda t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ . We now need to find a second solution. From a past homework assignment (i.e., see the solution to 5.2.5 from homework 4) we know that one candidate will be  $\mathbf{x} = de^{\lambda t} (t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \mathbf{y})$  where  $\mathbf{y}$  is a vector satisfying  $(A - \lambda I)\mathbf{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ . One such  $\mathbf{y}$  that will work (there are in fact many) is  $\begin{bmatrix} 0 \\ 1/b \end{bmatrix}$ . So we have that the general solution is

$$\mathbf{x} = ce^{\lambda t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + de^{\lambda t} \begin{bmatrix} t \\ 1/b \end{bmatrix}.$$

When it comes to sketching the phase portrait there are several possibilities depending on whether  $\lambda = -, 0, +$  and  $b = -, +$ . The key to all of them is noting that we know

explicitly what happens on the  $x$ -axis (i.e., the direction of the eigenvector) and then we just need to figure out how curves/trajectories come into/out of the degenerate node, which can be done by plotting a few points/trajectories. The different possibilities are shown in the phase portraits below.



\* Consider the following system.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 3 & 1 \\ 4 & 1 & -1 \\ 2 & 7 & -5 \end{bmatrix} \mathbf{x}$$

We solve this using the same approach as  $2 \times 2$  matrices, i.e., find the eigenvalues and eigenvectors. The only difference is that there is more bookkeeping involved. First to find the eigenvalues we find the characteristic polynomial, in this case

$$\begin{aligned} \det(xI - A) &= \det \left( \begin{bmatrix} x & -3 & -1 \\ -4 & x-1 & 1 \\ -2 & -7 & x+5 \end{bmatrix} \right) \\ &= x(x-1)(x+5) + 6 - 28 - 2(x-1) + 7x - 12(x+5) = x^3 + 4x^2 - 12x + 80. \end{aligned}$$

To solve a cubic we hope for the best in that there will be a “nice” solution (i.e., rational). For cubics with integer coefficients and a “1” in front of  $x^3$  then the possible nice roots are those which divide the constant term. In our case the candidates are  $\pm 1, \pm 2, \pm 4, \pm 5, \pm 8, \pm 10, \pm 16, \pm 20, \pm 40, \pm 80$ . We start going through these and find that 4 is a root. Now using that we do some long division (i.e., divide  $x - 4$  into

$x^3 + 4x^2 - 12x + 80$ ) and discover  $x^3 + 4x^2 - 12x + 80 = (x - 4)(x^2 + 8x + 20)$ . Finally to get the other roots we can use the quadratic equation so that the three roots are  $\lambda = 4, -4 - 2i, -4 + 2i$ .

The next step is to find the eigenvectors. For  $\lambda = 4$  we are looking for a vector  $\mathbf{x}$  so that  $(A - 4I)\mathbf{x} = 0$ , i.e., we want

$$\begin{bmatrix} -4 & 3 & 1 \\ 4 & -3 & -1 \\ 2 & 7 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is equivalent to the following three equations:

$$\begin{aligned} -4x_1 + 3x_2 + x_3 &= 0 \\ 4x_1 - 3x_2 - x_3 &= 0 \\ 2x_1 + 7x_2 - 9x_3 &= 0 \end{aligned}$$

If we now add twice the third equation to the first equation (eliminating the  $x_1$  term) we have  $17x_2 - 17x_3 = 0$  showing that  $x_2 = x_3$ . Putting this last relationship into any of the above equations we can conclude that  $x_1 = x_2$ . So we have that the eigenvector in this case is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

For  $\lambda = -4 + 2i$  we are looking for a vector  $\mathbf{x}$  so that  $(A - (-4 + 2i)I)\mathbf{x} = 0$ , i.e., we want

$$\begin{bmatrix} 4 - 2i & 3 & 1 \\ 4 & 5 - 2i & -1 \\ 2 & 7 & -1 - 2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is equivalent to the following three equations:

$$\begin{aligned} (4 - 2i)x_1 + 3x_2 + x_3 &= 0 \\ 4x_1 + (5 - 2i)x_2 - x_3 &= 0 \\ 2x_1 + 7x_2 + (-1 - 2i)x_3 &= 0 \end{aligned}$$

If we now add the first and second equation together (eliminating the  $x_3$  term) we have  $(8 - 2i)x_1 + (8 - 2i)x_2 = 0$  showing that  $x_2 = -x_1$ . Putting this last relationship into any of the above equations we can conclude that  $x_3 = (-1 + 2i)x_1$ . So we have that the eigenvector in this case is

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ -1 + 2i \end{bmatrix}.$$

For  $\lambda = -4 - 2i$  we can repeat the above process or just use the fact that it will be the complex conjugate of the eigenvector we just found so that the eigenvector in this case is

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ -1 - 2i \end{bmatrix}.$$

We now have that the general solution will be

$$\mathbf{x} = c_1 e^{4t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{(-4+2i)t} \begin{bmatrix} 1 \\ -1 \\ -1 + 2i \end{bmatrix} + c_3 e^{(-4-2i)t} \begin{bmatrix} 1 \\ -1 \\ -1 - 2i \end{bmatrix}.$$

However this solution is somewhat unsatisfying since it involves imaginary numbers. So we now work to rewrite it. One approach (a bit lengthy) is to do what we did in the previous homework (of course other approaches also work). Doing that we get the following:

$$\begin{aligned} & c_2 e^{(-4+2i)t} \begin{bmatrix} 1 \\ -1 \\ -1 + 2i \end{bmatrix} + c_3 e^{(-4-2i)t} \begin{bmatrix} 1 \\ -1 \\ -1 - 2i \end{bmatrix} \\ &= e^{-4t} \begin{bmatrix} c_2(\cos(2t) + i \sin(2t)) + c_3(\cos(2t) - i \sin(2t)) \\ -c_2(\cos(2t) + i \sin(2t)) - c_3(\cos(2t) - i \sin(2t)) \\ c_2(-1 + 2i)(\cos(2t) + i \sin(2t)) + c_3(-1 - 2i)(\cos(2t) - i \sin(2t)) \end{bmatrix} \\ &= e^{-4t} \begin{bmatrix} (c_2 + c_3) \cos(2t) + i(c_2 - c_3) \sin(2t) \\ -(c_2 + c_3) \cos(2t) - i(c_2 - c_3) \sin(2t) \\ (c_2 + c_3)(-\cos(2t) - 2 \sin(2t)) + i(c_2 - c_3)(2 \cos(2t) - \sin(2t)) \end{bmatrix} \\ &= \underbrace{(c_2 + c_3)}_{=C_2} e^{-4t} \begin{bmatrix} \cos(2t) \\ -\cos(2t) \\ -\cos(2t) - 2 \sin(2t) \end{bmatrix} + \underbrace{i(c_2 - c_3)}_{=C_3} e^{-4t} \begin{bmatrix} \sin(2t) \\ -\sin(2t) \\ 2 \cos(2t) - \sin(2t) \end{bmatrix} \end{aligned}$$

So that we have that the general solution is

$$\mathbf{x} = C_1 e^{4t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_2 e^{-4t} \begin{bmatrix} \cos(2t) \\ -\cos(2t) \\ -\cos(2t) - 2 \sin(2t) \end{bmatrix} + C_3 e^{-4t} \begin{bmatrix} \sin(2t) \\ -\sin(2t) \\ 2 \cos(2t) - \sin(2t) \end{bmatrix}.$$

To find the particular solution we need to find  $C_1, C_2, C_3$  so that

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

In other words we need to have

$$\begin{aligned} C_1 + C_2 &= 1 \\ C_1 - C_2 &= 0 \\ C_1 - C_2 + 2C_3 &= 0 \end{aligned}$$

Subtracting the second from the third shows that we need to have  $C_3 = 0$ . Adding the first two shows that  $2C_1 = 1$  so that  $C_1 = 1/2$  and the second shows that  $C_2 = C_1 = 1/2$ . So we have that the particular solution is

$$\begin{aligned} \mathbf{x} &= \frac{1}{2}e^{4t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2}e^{-4t} \begin{bmatrix} \cos(2t) \\ -\cos(2t) \\ -\cos(2t) - 2\sin(2t) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}e^{4t} + \frac{1}{2}e^{-4t}\cos(2t) \\ \frac{1}{2}e^{4t} - \frac{1}{2}e^{-4t}\cos(2t) \\ \frac{1}{2}e^{4t} - \frac{1}{2}e^{-4t}\cos(2t) - e^{-4t}\sin(2t) \end{bmatrix}. \end{aligned}$$

Now consider the following system.

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix} \mathbf{x}$$

First to find the eigenvalues we find the characteristic polynomial, in this case

$$\begin{aligned} \det(xI - A) &= \det \left( \begin{bmatrix} x-1 & 0 & -2 \\ 0 & x+1 & 2 \\ -2 & 2 & x \end{bmatrix} \right) \\ &= (x-1)(x+1)x + 0 + 0 - 4(x+1) - 4(x-1) = x^3 - 9x = x(x-3)(x+3). \end{aligned}$$

So we have that the eigenvalues are  $\lambda = -3, 0, 3$ .

The next step is to find the eigenvectors. For  $\lambda = -3$  we are looking for a vector  $\mathbf{x}$  so that  $(A + 3I)\mathbf{x} = 0$ , i.e., we want

$$\begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is equivalent to the following three equations:

$$\begin{aligned}4x_1 & & + 2x_3 & = 0 \\ & 2x_2 & - 2x_3 & = 0 \\ 2x_1 & - 2x_2 & + 3x_3 & = 0\end{aligned}$$

The second equation tells us that  $x_2 = x_3$  while the first equation tells us that  $x_3 = -2x_1$ . So we have that the eigenvector in this case is

$$\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}.$$

For  $\lambda = 0$  we are looking for a vector  $\mathbf{x}$  so that  $A\mathbf{x} = 0$ , i.e., we want

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is equivalent to the following three equations:

$$\begin{aligned}x_1 & & + 2x_3 & = 0 \\ & - x_2 & - 2x_3 & = 0 \\ 2x_1 & - 2x_2 & & = 0\end{aligned}$$

The first equation tells us that  $x_1 = -2x_3$  while the second equation tells us that  $x_2 = -2x_3$ . So we have that the eigenvector in this case is

$$\mathbf{x} = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}.$$

For  $\lambda = 3$  we are looking for a vector  $\mathbf{x}$  so that  $(A - 3I)\mathbf{x} = 0$ , i.e., we want

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & -2 \\ 2 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is equivalent to the following three equations:

$$\begin{aligned}-2x_1 & & + 2x_3 & = 0 \\ & -4x_2 & - 2x_3 & = 0 \\ 2x_1 & - 2x_2 & - 3x_3 & = 0\end{aligned}$$

The first equation tells us that  $x_1 = x_3$  while the second equation tells us that  $x_3 = -2x_2$ . So we have that the eigenvector in this case is

$$\mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}.$$

We now have that the general solution will be

$$\mathbf{x} = c_1 e^{-3t} \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}.$$

To find the particular solution we need to find  $c_1, c_2, c_3$  so that

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

In other words we need to have

$$\begin{aligned} c_1 - 2c_2 - 2c_3 &= 1 \\ -2c_1 - 2c_2 + c_3 &= 0 \\ -2c_1 + c_2 - 2c_3 &= 0 \end{aligned}$$

If we now double the first equation and add it to the second and third equations this becomes

$$\begin{aligned} c_1 - 2c_2 - 2c_3 &= 1 \\ - 6c_2 - 3c_3 &= 2 \\ - 3c_2 - 6c_3 &= 2 \end{aligned}$$

If we now multiply the second equation by  $-2$  and add it to the third equation this becomes

$$\begin{aligned} c_1 - 2c_2 - 2c_3 &= 1 \\ - 6c_2 - 3c_3 &= 2 \\ 9c_2 &= -2 \end{aligned}$$

It immediately follows that  $c_2 = -2/9$  substituting this into the second equation we see that  $c_3 = -2/9$  then substituting these values into the first equation we see that  $c_1 = 1/9$ . So we have that the particular solution is

$$\begin{aligned} \mathbf{x} &= \frac{1}{9} e^{-3t} \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} - \frac{2}{9} \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} - \frac{2}{9} e^{3t} \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} \\ &= \frac{1}{9} \begin{bmatrix} e^{-3t} + 4 + 4e^{3t} \\ -2e^{-3t} + 4 - 2e^{3t} \\ -2e^{-3t} - 2 + 4e^{3t} \end{bmatrix}. \end{aligned}$$