# Homework 6 solutions

(1) (a) We have that the characteristic equation for the matrix is

$$\det\left(\begin{bmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix}\right) = (2-\lambda)(-1-\lambda) - 4 = \lambda^2 - \lambda - 6 = (\lambda-3)(\lambda+2),$$

so that the eigenvalues are 3 and $-2$. To find an eigenvector for 3 we can take a column of $A+2I$, so $\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$, and to find an eigenvector for $-2$ we can take a column of $A-3I$, so $\left[\begin{smallmatrix} -1 \\ 2 \end{smallmatrix}\right]$. We now have that

$$\underbrace{\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}}_{=A}\underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}}_{=S} = \underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}}_{=S}\underbrace{\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}}_{=\Lambda}.$$

The next step is to find the inverse of $S$, which can be done by some row reduction such as follows:

$$\left[\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{5}{2} & -\frac{1}{2} & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{array}\right]$$

So we have that

$$S^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}.$$

We now have that $e^{At} = Se^{\Lambda t}S^{-1}$ and since $\Lambda$ is diagonal this is easy to compute. In particular, we have

$$e^{At} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4e^{3t}+e^{-2t} & 2e^{3t}-2e^{-2t} \\ 2e^{3t}-2e^{-2t} & e^{3t}+4e^{-2t} \end{bmatrix}.$$

(b) We have that the characteristic equation for the matrix is

$$\det\left(\begin{bmatrix} 1-\lambda & 2 \\ -5 & -1-\lambda \end{bmatrix}\right) = (1-\lambda)(-1-\lambda) + 10 = \lambda^2 + 9,$$

so that the eigenvalues are $3i$ and $-3i$. To find an eigenvector for $3i$ we can take a column of $A+(3i)I$, so $\begin{bmatrix} 1+3i \\ -5 \end{bmatrix}$, and to find an eigenvector for $-3i$ we can take a column of $A-(3i)I$, so $\begin{bmatrix} 1-3i \\ -5 \end{bmatrix}$. We now have that

$$\underbrace{\begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}}_{=A} \underbrace{\begin{bmatrix} 1+3i & 1-3i \\ -5 & -5 \end{bmatrix}}_{=S} = \underbrace{\begin{bmatrix} 1+3i & 1-3i \\ -5 & -5 \end{bmatrix}}_{=S} \underbrace{\begin{bmatrix} 3i & 0 \\ 0 & -3i \end{bmatrix}}_{=\Lambda}.$$

The next step is to find the inverse of $S$, this can be done using the same procedure as in part (a) or letting a computer do it. In either case we have

$$S^{-1} = \frac{1}{30} \begin{bmatrix} -5i & -3-i \\ 5i & -3+i \end{bmatrix}.$$

We now have that $e^{At} = Se^{\Lambda t}S^{-1}$ and since $\Lambda$ is diagonal this is easy to compute. In particular, we have after multiplying and a lot of simplifying

$$\begin{aligned} e^{At} &= \frac{1}{30} \begin{bmatrix} 1+3i & 1-3i \\ -5 & -5 \end{bmatrix} \begin{bmatrix} \cos(3t) + i\sin(3t) & 0 \\ 0 & \cos(3t) - i\sin(3t) \end{bmatrix} \begin{bmatrix} -5i & -3-i \\ 5i & -3+i \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 3\cos(3t) + \sin(3t) & 2\sin(3t) \\ -5\sin(3t) & 3\cos(3t) - \sin(3t) \end{bmatrix}. \end{aligned}$$

(c) We clearly have that the eigenvalues for $A$ are 2 and $2+\epsilon$. Further an eigenvector for 2 is a column of $A-(2+\epsilon)I$, so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and an eigenvector for $2+\epsilon$ is a column of $A-2I$ so $\begin{bmatrix} 1 \\ \epsilon \end{bmatrix}$. So we have that

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 2+\epsilon \end{bmatrix}}_{=A} \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & \epsilon \end{bmatrix}}_{=S} = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & \epsilon \end{bmatrix}}_{=S} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 2+\epsilon \end{bmatrix}}_{=\Lambda}.$$

Similar as in part (a) we now need to find $S^{-1}$ which can be done using row reduction to get:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & \epsilon & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{\epsilon} \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & -\frac{1}{\epsilon} \\ 0 & 1 & 0 & \frac{1}{\epsilon} \end{array}\right].$$

So we have that

$$S^{-1} = \begin{bmatrix} 1 & -\frac{1}{\epsilon} \\ 0 & \frac{1}{\epsilon} \end{bmatrix}.$$

We now have that $e^{At} = Se^{\Lambda t}S^{-1}$ and since $\Lambda$ is diagonal this is easy to compute. In particular, we have after multiplying that

$$e^{At} = \begin{bmatrix} 1 & 1 \\ 0 & \epsilon \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{(2+\epsilon)t} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{\epsilon} \\ 0 & \frac{1}{\epsilon} \end{bmatrix} = \begin{bmatrix} e^{2t} & \dfrac{e^{(2+\epsilon)t} - e^{2t}}{\epsilon} \\ 0 & e^{(2+\epsilon)t} \end{bmatrix}.$$

If we now let $\epsilon \to 0$ then both the diagonal terms go to $e^{2t}$. For the upper right entry the limit tends to $\frac{0}{0}$ and so we can use L'Hospital's rule (here remembering that we are taking the derivative with respect to $\epsilon$ and not $t$ to get that

$$\lim_{\epsilon \to 0} \frac{e^{(2+\epsilon)t} - e^{2t}}{\epsilon} = \lim_{\epsilon \to 0} \frac{te^{(2+\epsilon)t}}{1} = te^{2t}.$$

So the matrix as $\epsilon \to 0$ tends to

$$\begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}.$$

(2) Starting with

$$\dot{\mathbf{x}} = \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}}_{=A} \mathbf{x} + \underbrace{\begin{bmatrix} e^{2t} \\ te^{2t} \end{bmatrix}}_{=\mathbf{y}},$$

we can rearrange this as $\dot{\mathbf{x}} - A\mathbf{x} = \mathbf{y}$. We can then use the same integrating factor techniques as we did in homework 0 to get that

$$e^{-At}\dot{\mathbf{x}} - Ae^{-At}\mathbf{x} = \left(e^{-At}\mathbf{x}\right)' = e^{-At}\mathbf{y}$$

$$\text{so that} \quad e^{-At}\mathbf{x} = \int e^{-At}\mathbf{y}\,dt \quad \text{or} \quad \mathbf{x} = e^{At}\int e^{-At}\mathbf{y}\,dt$$

From the last problem we know what $e^{At}$ is. To find $e^{-At}$ we can repeat a similar process or note that $e^{-At} = (e^{At})^{-1}$, i.e., is the inverse of $e^{At}$. So finding inverses we have

$$\left[\begin{array}{cc|cc} e^{2t} & te^{2t} & 1 & 0 \\ 0 & e^{2t} & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & t & e^{-2t} & 0 \\ 0 & 1 & 0 & e^{-2t} \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & e^{-2t} & -te^{-2t} \\ 0 & 1 & 0 & e^{-2t} \end{array}\right].$$

So we have that

$$e^{-At} = \begin{bmatrix} e^{-2t} & -te^{-2t} \\ 0 & e^{-2t} \end{bmatrix}.$$

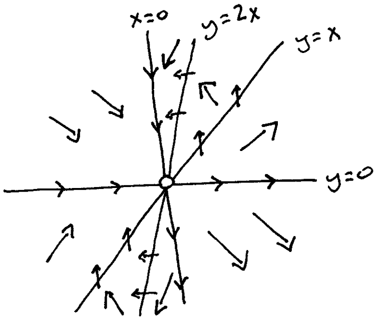
Plugging this in we have that

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix} \int \begin{bmatrix} e^{-2t} & -te^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} e^{2t} \\ te^{2t} \end{bmatrix} dt \\ &= \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix} \int \begin{bmatrix} 1 - t^2 \\ t \end{bmatrix} dt \\ &= \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} t - \frac{1}{3}t^3 + c \\ \frac{1}{2}t^2 + d \end{bmatrix} \\ &= e^{2t} \begin{bmatrix} \frac{1}{6}t^3 + t + dt + c \\ \frac{1}{2}t^2 + d \end{bmatrix}. \end{aligned}$$

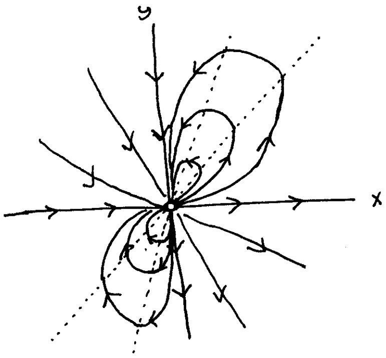
Our initial condition is that $\mathbf{x}(0) = \left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$ from which it follows that $c = -1$ and $d = 1$ so that the final solution to the initial value problem is

$$\mathbf{x} = e^{2t} \begin{bmatrix} \frac{1}{6}t^3 + 2t - 1 \\ \frac{1}{2}t^2 + 1 \end{bmatrix}.$$

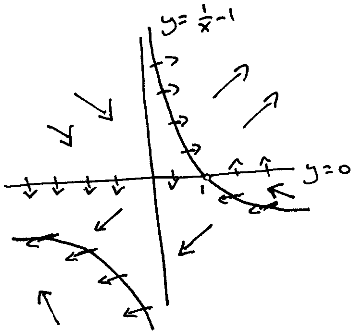
6.1.3 To find the fixed point we need to have $x(x - y) = 0$ and $y(2x - y) = 0$. The first equation says that $x = 0$ or $x = y$ while the second equation tells us that $y = 0$ or $2x = y$; taking any combination of these equations gives us the unique fixed point $(x^*, y^*) = (0, 0)$. The nullclines for which $\dot{x} = 0$ are $x = 0$ and $y = x$ while the nullclines for which $\dot{y} = 0$ are $y = 0$ and $y = 2x$. This gives us the following sketch for the nullclines (where we have also indicated the direction of the flow):



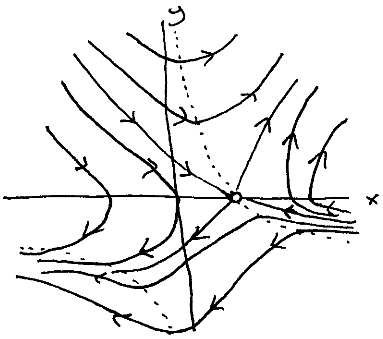
From which we can get the following rough sketch of the phase portrait:



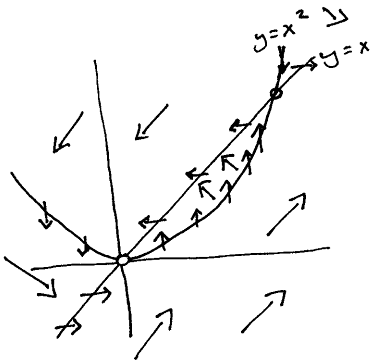
6.1.4 To find the fixed point we need to have $y = 0$ and $x(1 + y) = 1$. This gives us the unique fixed point $(x^*, y^*) = (1, 0)$. The nullcline for which $\dot{x} = 0$ is $y = 0$ while the nullclines for which $\dot{y} = 0$ is $y = \frac{1}{x} - 1$. This gives us the following sketch for the nullclines (where we have also indicated the direction of the flow):



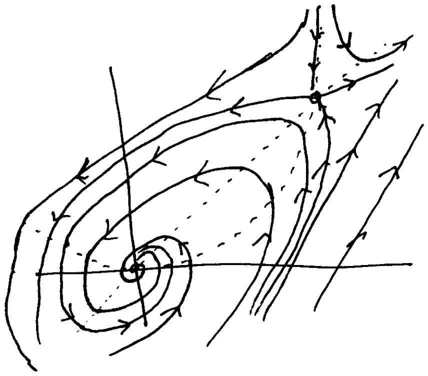
From which we can get the following rough sketch of the phase portrait:



6.1.6 To find the fixed point we need to have $y = x^2$ and $y = x$. Combining the two we have that $x^2 = x$ so that $x = 0$ or $x = 1$ giving us two different fixed points, namely $(x^*, y^*) = (0, 0)$ and $(x^*, y^*) = (1, 1)$. The nullcline for which $\dot{x} = 0$ are $y = x^2$ and $y = x$ while the nullcline for which $\dot{y} = 0$ are $y = x$. This gives us the following sketch for the nullclines (where we have also indicated the direction of the flow):



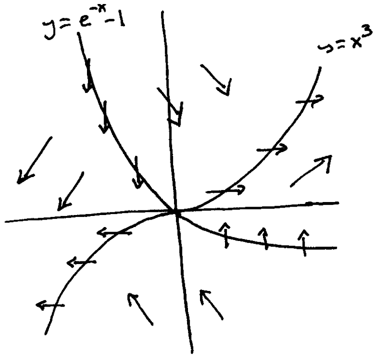
From which we can get the following rough sketch of the phase portrait:



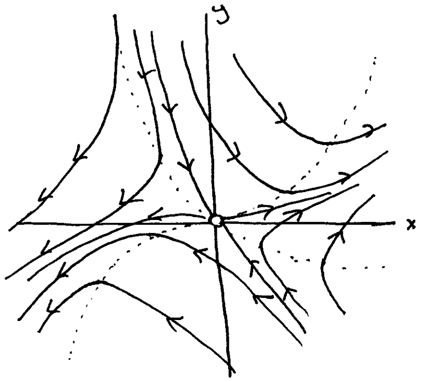
6.3.3 Starting with $\dot{x} = 1 + y - e^{-x}$ and $\dot{y} = x^3 - y$ a fixed point will occur at the intersection of the curves $y = e^{-x} - 1$ and $y = x^3$. It is simple to see (i.e., such as by graphing) that the only intersection of these curves is at $(x^*, y^*) = (0, 0)$. We now linearize around $(0, 0)$ to get the following system

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \mathbf{x},$$

since the determinant is negative then the eigenvalues are real and have different signs and so this is a saddlepoint. If we sketch the nullclines and indicate the directions we get the following:



From which we can get the following rough sketch of the phase portrait:



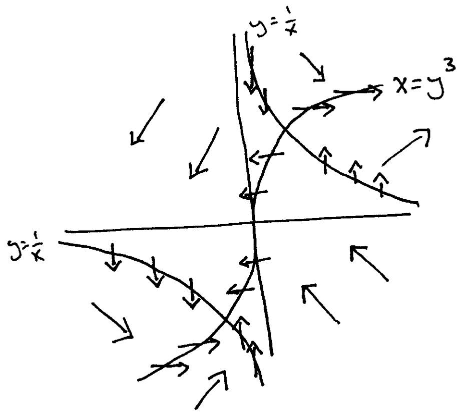
6.3.6 Starting with $\dot{x} = xy - 1$ and $\dot{y} = x - y^3$ a fixed point will occur at the intersection of the curves $x = \frac{1}{y}$ and $x = y^3$. It is simple to see (i.e., such as by graphing or algebra) that the only intersections of these curves are at $(x^*, y^*) = (1, 1)$ and $(-1, -1)$. We now linearize around $(1, 1)$ to get the following system

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{x},$$

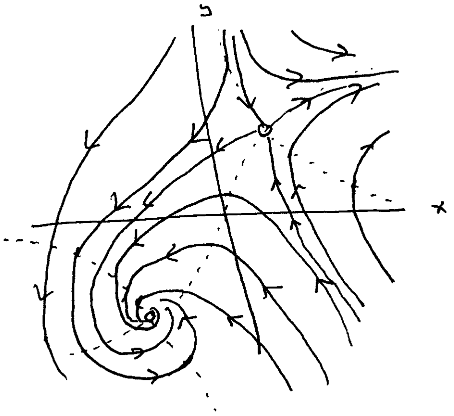
since the determinant is negative then the eigenvalues are real and have different signs and so this is a saddlepoint. If we now linearize around $(-1, -1)$ to get the following system

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} \mathbf{x},$$

which has a repeated eigenvalue of $-2$ but is deficient. Because of the deficiency we cannot determine what happens, based on the picture below it appears to be a spiral. If we sketch the nullclines and indicate the directions we get the following:



From which we can get the following rough sketch of the phase portrait:



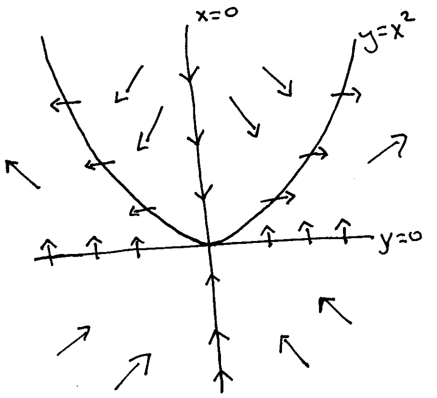
6.3.10 (a) Starting with $\dot{x} = xy$ and $\dot{y} = x^2 - y$ the linearization gives

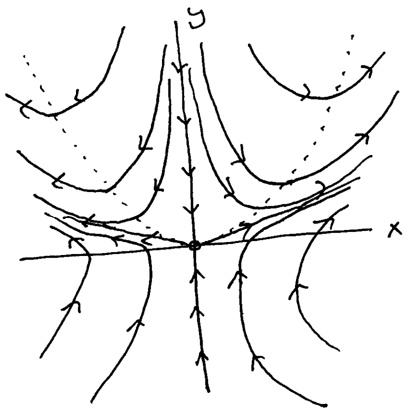
$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}.$$

This has eigenvalues of 0 and $-1$ and since 0 is an eigenvalue this predicts non-isolated fixed points.

(b) To find the fixed points of the system we have that $xy = 0$ and $x^2 - y = 0$, multiplying both sides of the second equation by $x$ we have $x^3 - xy = 0$, but since $xy = 0$ this reduces to $x^3 = 0$ or $x = 0$, from which it follows that $y = 0$. In particular there is a unique isolated fixed point at $(x^*, y^*) = (0, 0)$.

(c) It is easy to find the nullclines ($x = 0$, $y = 0$ and $y = x^2$) so that we can get a rough sketch of the vector field and the phase portrait as follows:





In particular we see that the origin is a saddle.