Midterm 1 solutions

(1) Consider the differential equation $\dot{x} = x^4$.

(a) If we specify the value of $x(0)$, is the solution unique? How do you know?

SOLUTION: Since $f(x) = x^4$ and $f'(x) = 4x^3$ are continuous in some interval around 0 then the *existence and uniqueness theorem* assures that for a specified initial value a unique solution exists.

(b) What is the solution if $x(0) = 2$? Does this solution blow up at any finite time $t > 0$?

SOLUTION: Using separation of variables we have the following:

$$\begin{aligned} \dot{x} = x^4 \quad &\Rightarrow \quad \frac{dx}{dt} = x^4 \\ &\Rightarrow \quad \frac{dx}{x^4} = dt \\ &\Rightarrow \quad \int \frac{dx}{x^4} = \int dt \\ &\Rightarrow \quad \frac{-1}{3x^3} = t + C. \end{aligned}$$

Plugging in the initial condition $x(0) = 2$ we see that $C = -1/24$. So we have that

$$\frac{-1}{3x^3} = t - \frac{1}{24} \quad \Rightarrow \quad x^3 = \frac{1}{\frac{1}{8} - 3t} \quad \Rightarrow \quad x = \frac{1}{\sqrt[3]{\frac{1}{8} - 3t}}.$$

This function would blow up if the denominator went to 0. So in our case it is simple to see that at $t = 1/24$ the function blows up, i.e., it *does* blow up in finite time.

(c) What is the solution if $x(0) = -2$? Does it reach $x = 0$ at any finite time $t > 0$?

SOLUTION: Using the same approach as in part (b) we have that

$$\frac{-1}{3x^3} = t + C.$$

Now if we plug in the initial condition $x(0) = -2$ we see that $C = 1/24$. So we have that

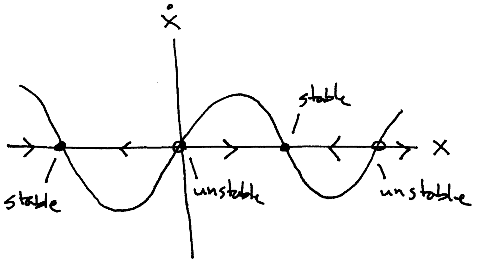
$$\frac{-1}{3x^3} = t + \frac{1}{24} \quad \Rightarrow \quad x^3 = \frac{-1}{\frac{1}{8} + 3t} \quad \Rightarrow \quad x = \frac{-1}{\sqrt[3]{\frac{1}{8} + 3t}}.$$

Clearly for any time $t > 0$ the denominator is a finite nonzero number and so $x(t) \neq 0$ for all t, i.e., it does not reach the origin in finite time.

(2) Consider the differential equation $\dot{x} = r + \sin x$, where r is a real parameter.

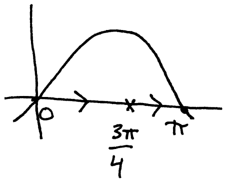
(a) Sketch the phase portrait for this equation if $r = 0$. Indicate the direction of the vector field and the stability of any fixed points. If $x(0) = 3\pi/4$, find $\lim_{t\to\pm\infty} x(t)$.

SOLUTION: The sketch of the phase portrait is shown below and we have also indicated the direction of the vector field and the stability of the first few fixed points.



In general the fixed points will occur at the integer multiples of π with the odd integer multiples stable and even integer multiples unstable.

To examine what happens if $x(0) = 3\pi/4$ let us look at that portion of the above figure more closely.

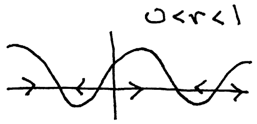


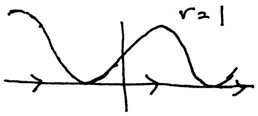
We see that as $t \to +\infty$ that the value of x goes to the stable fixed point at π (i.e., x moves to the stable fixed point). On the other hand as $t \to -\infty$ the value of x will go to the unstable fixed point at 0 (i.e., as $t \to -\infty$ we are trying to figure out where the particle would be coming from). So we have that

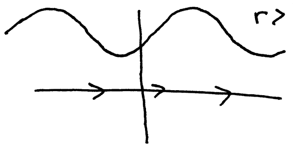
$$\lim_{t\to+\infty} x(t) = \pi \qquad \text{and} \qquad \lim_{t\to-\infty} x(t) = 0.$$

(b) Now suppose that $r > 0$. How does the phase portrait change as r increases? Determine any values for r where the bifurcations occur, and the type of bifurcation. Sketch the bifurcation diagram.

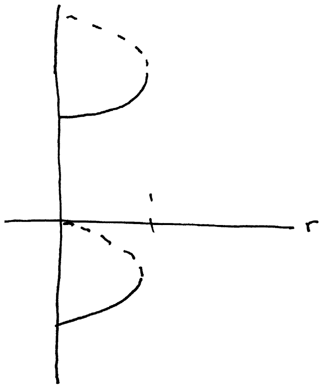
Solution: When $0 < r < 1$ we have shifted the sine curve up only a little bit and so the curve still crosses through the axis and we still have the periodic alternating stable and unstable points. When $r = 1$ we will have shifted the curve up so that at $3\pi/2 + 2\pi k$ the curve just touches the axis, in particular we now have a lot of semistable points. When $r > 1$ the sine curve has been shifted entirely above the axis and there will no longer be any fixed points. Examples of these three possibilities are shown below.







From this it follows that our bifurcation occurs at $r = 1$ (and $x = 3\pi/2 + 2\pi k$) and all the bifurcations are saddle node bifurcations. It is now easy to use the above discussions and illustrations to draw the following bifurcation diagram (note that the curves are the portions of $r = -\sin x$ that also satisfy $r > 0$).



(3) The velocity $v(t) > 0$ of an object falling through air obeys the differential equation $m\dot{v} = mg - kv^2$, where m is the object's mass, g is the (constant) gravitational acceleration, and k is a constant measuring the air resistance.

(a) Find the equilibrium velocity, also called the terminal velocity.

Solution: The equilibrium velocity refers to the velocity with $\dot{v} = 0$. Plugging this into our differential equation we have $0 = mg - kv^2$ or $v^2 = mg/k$ or $v = \sqrt{mg/k}$ (here we do not use the "±" since we assume $v > 0$).

(b) Introduce a new variable $u(t)$ which is the velocity measured in units of the terminal velocity and derive the differential equation obeyed by $u(t)$.

Solution: To measure velocity in units of terminal velocity we divide the velocity by the terminal velocity, i.e., $u(t) = v(t)/\sqrt{mg/k}$, or $v(t) = \sqrt{mg/k}\,u(t)$. From this it follows that $\dot{v} = \sqrt{mg/k}\,\dot{u}$. If we now substitute this into $m\dot{v} = mg - kv^2$ we have

$$m\sqrt{\frac{mg}{k}}\,\dot{u} = mg - k\big(\frac{mg}{k}u^2\big) \qquad \text{or} \qquad \sqrt{\frac{m}{kg}}\,\dot{u} = 1 - u^2.$$

(c) Introduce a new time variable τ in such a way that $du/d\tau = f(u)$, where $f(u)$ contains none of the parameters m, g, k.

Solution: We want the following

$$\sqrt{\frac{m}{kg}}\,\underbrace{\frac{du}{dt}}_{=\dot{u}} = \frac{du}{d\tau} = \underbrace{\frac{du}{dt}\frac{dt}{d\tau}}_{\text{chain rule}}\,.$$

In particular we need to have $dt/d\tau = \sqrt{m/kg}$ or $dt = \sqrt{m/kg}\,d\tau$. Integrating both sides we see that we need $t = \sqrt{m/kg}\,\tau$, or equivalently $\tau = \sqrt{kg/m}\,t$. It now follows that we have

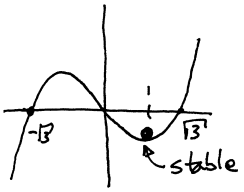
$$\frac{du}{d\tau} = 1 - u^2.$$

(d) Give a potential energy function $V(u)$ for the equation in part (c). Use it to show that the fixed point $u = 1$ is stable.

Solution: From the last part we have

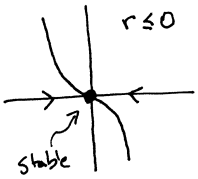
$$\frac{du}{d\tau} = 1 - u^2 = -V'(u),$$

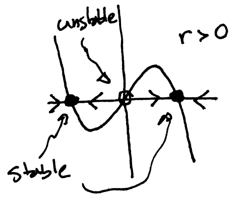
so that $V'(u) = u^2 - 1$ giving a potential energy function of $V(u) = \frac{1}{3}u^3 - u$. This potential energy function is easy to graph and we show it below, in particular we see that there is a min at $u = 1$ and so this will correspond to a stable point.



(4) Draw the bifurcation diagram for the equation $\dot{x} = rx - x^5$, where r is a parameter. The result should be similar to a type of bifurcation we studied in class; which one?

Solution: First we note that $\dot{x} = rx - x^5 = x(r - x^4)$ and so the zeroes will correspond to when either $x = 0$ or $r = x^4$, i.e., $x = \pm\sqrt[4]{r}$. In particular for $r \leq 0$ there will only be a single root at 0 and for $r \geq 0$ there will be three roots at $-\sqrt[4]{r}, 0, \sqrt[4]{r}$. If we draw pictures for these two cases we get the following





Alternatively, we can use linear stability analysis since if $f(x) = rx - x^5$ then $f'(x) = r - 5x^4$ so that for $x = 0$ we see that $f'(x) = r$ which is stable if $r < 0$ and unstable if $r > 0$ and for $x^4 = r$ this becomes $f'(x) = -4r$ which will be stable (since again for r negative there are no solutions for x).

In particular we get the following bifurcation diagram which is similar to the *pitchfork* bifurcation that we studied.