Midterm 2 solutions

(1) Consider the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ with the matrix

$$A = \begin{bmatrix} 6 & -9 \\ 2 & -1 \end{bmatrix}.$$

(a) Find the general solution.

Solution: We start by finding the eigenvalues and eigenvectors. We have that the characteristic equation is

$$\det\left(\begin{bmatrix} 6-\lambda & -9 \\ 2 & -1-\lambda \end{bmatrix}\right) = (6-\lambda)(-1-\lambda) - 18 = \lambda^2 - 5\lambda - 24 = (\lambda-8)(\lambda+3).$$

So the eigenvalues are 8 and -3. To find the eigenvectors we can use the homework problem which says that an eigenvector for 8 is a column of the matrix $A - (-3)I$, so $\left[\begin{smallmatrix} 9 \\ 2 \end{smallmatrix}\right]$, and an eigenvector for -3 is a column of the matrix $A - 8I$, so $\left[\begin{smallmatrix} -2 \\ 2 \end{smallmatrix}\right]$ or scaling $\left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$.
So we now have that the general solution is

$$\mathbf{x} = ce^{8t}\begin{bmatrix} 9 \\ 2 \end{bmatrix} + de^{-3t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(b) Find the particular solution sastisfying $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$.

Solution: Plugging in 0 to the solution in part (a) we need to find c and d so that

$$\begin{bmatrix} 0 \\ 11 \end{bmatrix} = c\begin{bmatrix} 9 \\ 2 \end{bmatrix} + d\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

This is equivalent to solving the following system of linear equations:

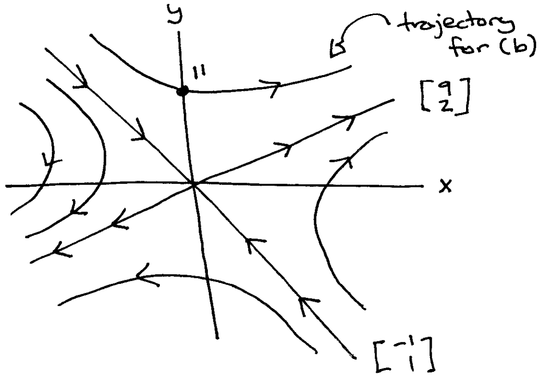
$$\begin{array}{ccccc} 9c & - & d & = & 0 \\ 2c & + & d & = & 11 \end{array}$$

If we add these together we see that $11c = 11$ or $c = 1$ from which it follows that $d = 9$. So we have that the particular solution is

$$\mathbf{x} = e^{8t}\begin{bmatrix} 9 \\ 2 \end{bmatrix} + 9e^{-3t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(c) Sketch the phase portrait for this system showing several trajectories and labeling the one that represents your solution to part (b). Include the eigenvectors in your sketch if they are real. What kind of fixed point is the origin for this system?

SOLUTION: Since we have one positive and negative root then the origin will be a saddle. To sketch it we first draw the eigenvectors and then some additional trajectories (marking directions). Finally the solution to part (b) will pass through $x = 0$ and $y = 11$ making it easy to mark that trajectory.



(2) The differential equation $\ddot{x} + 2\dot{x} + 3x = 0$ describes the position $x(t)$ of a harmonic oscillator with friction.

(a) Convert this second-order equation to a linear system of two first-order equations by introducing a second function $y = \dot{x}$.

SOLUTION: We have $\dot{x} = y$, on the other hand we have $\dot{y} = \ddot{x}$ and since $\ddot{x} + 2\dot{x} + 3x = 0$ we have $\dot{y} = -2\dot{x} - 3x = -2y - 3x$. So we have the following system of equations:

$$\begin{array}{rcrcr} \dot{x} & = & & & y \\ \dot{y} & = & -3x & - & 2y \end{array}$$

(b) Find two linearly independent real solutions of the system.

SOLUTION: Writing the system in matrix form we have

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}.$$

Finding the characteristic equation we get

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -3 & -2-\lambda \end{bmatrix}\right) = -\lambda(-2-\lambda) + 3 = \lambda^2 + 2\lambda + 3.$$

Using the quadratic formula to find roots we see that the eigenvalues are $-1\pm\sqrt{2}i$. To find the eigenvector for $-1+\sqrt{2}i$ we can take a column of $A-(-1-\sqrt{2}i)I$, so $\begin{bmatrix}1+\sqrt{2}i\\-3\end{bmatrix}$, and similarly to find the eigenvector for $-1-\sqrt{2}i$ we can take a column of $A-(-1+\sqrt{2}i)I$, so $\begin{bmatrix}1-\sqrt{2}i\\-3\end{bmatrix}$.

We now have that a general (but not real) solution is

$$\mathbf{x} = ce^{(-1+\sqrt{2}i)t}\begin{bmatrix}1+\sqrt{2}i\\-3\end{bmatrix} + de^{(-1-\sqrt{2}i)t}\begin{bmatrix}1-\sqrt{2}i\\-3\end{bmatrix}.$$

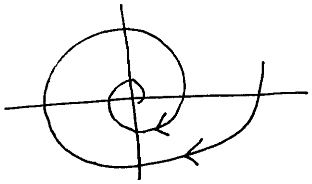
To get real solutions we have to combine these solutions, alternatively we can take the real and imaginary parts of one of these solutions, i.e.,

$$\begin{aligned}e^{(-1+\sqrt{2}i)t}\begin{bmatrix}1+\sqrt{2}i\\-3\end{bmatrix} &= e^{-t}\begin{bmatrix}(\cos(\sqrt{2}t)+i\sin(\sqrt{2}t))(1+\sqrt{2}i)\\-3(\cos(\sqrt{2}t)+i\sin(\sqrt{2}t))\end{bmatrix}\\ &= \underbrace{e^{-t}\begin{bmatrix}\cos(\sqrt{2}t)-\sqrt{2}\sin(\sqrt{2}t)\\-3\cos(\sqrt{2}t)\end{bmatrix}}_{=\text{ first real solution}} + i\,\underbrace{e^{-t}\begin{bmatrix}\sqrt{2}\cos(\sqrt{2}t)+\sin(\sqrt{2}t)\\-3\sin(\sqrt{2}t)\end{bmatrix}}_{=\text{ second real solution}}.\end{aligned}$$

(The solutions to this might vary depending on which eigenvectors were used.)

(c) Sketch a typical trajectory for the system.

SOLUTION: Since the eigenvalues are complex and the real part is negative this forms a stable spiral. To determine if we spiral clockwise or counterclockwise we can determine how it crosses the positive x-axis as t is increasing, using the above matrix with $\mathbf{x}=\begin{bmatrix}1\\0\end{bmatrix}$ we see that $\dot{y}<0$ so that it crosses in a downward direction, giving us the general shape below.



(3) Consider the flow on the circle given by $\dot{\theta}=(\frac{1}{2}+\sin\theta)^2+b$, where b is a real parameter.

(a) For what values of b does the system have periodic solutions?

SOLUTION: To be periodic means that the solution θ repeats, i.e., we can get all the way around. The only way that this is possible is if either $\dot{\theta}>0$ for all

θ in which case we circle around counter-clockwise or if $\dot{\theta} < 0$ for all θ in which case we circle around clockwise. The only way that we do not have a periodic solution is if there are fixed points. In particular this question can be rephrased as for what values of b does θ have no fixed points.

First note that $0 \leq (\frac{1}{2} + \sin\theta)^2 \leq \frac{9}{4}$ so that $b \leq (\frac{1}{2} + \sin\theta)^2 + b \leq \frac{9}{4} + b$. So now for $\dot{\theta} < 0$ for all θ we need $\frac{9}{4} + b < 0$, i.e., $b < -\frac{9}{4}$ while for $\dot{\theta} > 0$ for all θ we need $b > 0$.

So there will be a periodic solution if and only if $b > 0$ or $b < -\frac{9}{4}$.

(b) Set up but do not evaluate a definite integral giving the period of a periodic solution.

SOLUTION: As a general rule if $\dot{\theta} = f(\theta)$ has a periodic solution the the period is found by $\int_0^{2\pi} \frac{d\theta}{f(\theta)}$. So in this case we have,

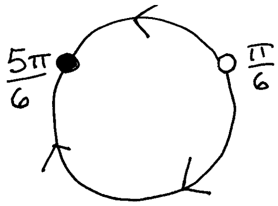
$$\text{period} = \int_0^{2\pi} \frac{d\theta}{(\frac{1}{2} + \sin\theta)^2 + b}.$$

(c) Locate the fixed points, determine their stability, and sketch a phase portrait in case $b = -1$.

SOLUTION: For $\dot{\theta} = (\frac{1}{2} + \sin\theta)^2 - 1$ the fixed points will occur when $(\frac{1}{2} + \sin\theta)^2 - 1 = 0$. This is equivalent to $(\frac{1}{2} + \sin\theta)^2 = 1$, or $\frac{1}{2} + \sin\theta = \pm 1$, or $\sin\theta = \frac{1}{2}, -\frac{3}{2}$. Of course $-\frac{3}{2}$ is not possible, but $\sin\theta = \frac{1}{2}$ has solutions at $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

To determine stability we can use (among other options) linear stability analysis, i.e., if $f(\theta) = (\frac{1}{2} + \sin\theta)^2 - 1$ then $f'(\theta) = 2(\frac{1}{2} + \sin\theta)\cos\theta$. In particular, $f'(\frac{\pi}{6}) = \sqrt{3} > 0$ and so $\frac{\pi}{6}$ is unstable, while $f'(\frac{5\pi}{6}) = -\sqrt{3} < 0$ and so $\frac{5\pi}{6}$ is stable.

Having found the fixed points and stability we can easily sketch the phase portrait as below.



(4) Consider the 2×2 linear system with the matrix

$$A = \begin{bmatrix} 1 & b \\ 2 & 3 \end{bmatrix},$$

where b is a real parameter. As b varies, the eigenvalues and therefore the type of fixed point may vary also. This is how bifurcations occur in linear systems. Determine the specific values of b where the type of fixed point changes, and classify the types of fixed point which occur at all other values of b. Hint: varying b also changes the trace and determinant of A.

SOLUTION: The type of node is determined by the eigenvalues. Calculating we have that the characteristic equation is

$$\det\left(\begin{bmatrix} 1-\lambda & b \\ 2 & 3-\lambda \end{bmatrix}\right) = (1-\lambda)(3-\lambda) - 2b = \lambda^2 - 4\lambda + (3-2b).$$

Using the quadratic formula we have that the eigenvalues are

$$\frac{4 \pm \sqrt{16 - 4(3-2b)}}{2} = 2 \pm \sqrt{2b+1}.$$

Whenever there is a square root the first thing to check is if the term in the square root is positive or negative (i.e., giving real or complex roots), here the dividing line will occur when $2b+1=0$, i.e., $b = -\frac{1}{2}$. On the other hand if we have real roots then we can either have both positive roots or one positive and one negative (we cannot have both negative in our case since one eigenvalue is 2+(something)), it is easy to see that the division between these two cases will occur when one of our eigenvalues is 0 and this will happen at $b = \frac{3}{2}$.

So we now have the following:

* $b < -\frac{1}{2}$: complex eigenvalues with positive real part, giving an unstable spiral.
* $b = -\frac{1}{2}$: repeated eigenvalue and in this case a degenerate node.
* $-\frac{1}{2} < b < \frac{3}{2}$: two distinct positive eigenvalues, giving an unstable node.
* $b = \frac{3}{2}$: one positive, one zero eigenvalue, giving non-isolated fixed points.
* $\frac{3}{2} < b$: one positive, one negative eigenvalue, giving a saddle node.

So there are bifurcations at $-\frac{1}{2}$ and $\frac{3}{2}$.