Items to review

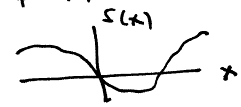
200 {
* separation of variables ((1) separate, (2) integrate, (3) simplify)
* integrating factor ($\dot{x} + f(t)x = g(t)$ multiply both sides by $e^{\int f(t)dt}$ so LHS becomes $(e^{\int f(t)dt}x)'$)

1-dimensional

* Fixed pts ($\dot{x} = f(x)$, fixed points occur at $f(x) = 0$)
* stability

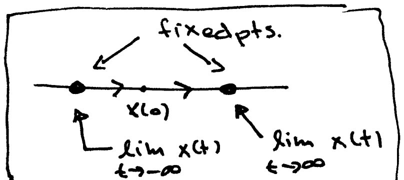
stable (move toward) — unstable (move away) — semistable (different behavior on bothsides) or

for $\dot{x} = f(x)$ plot:



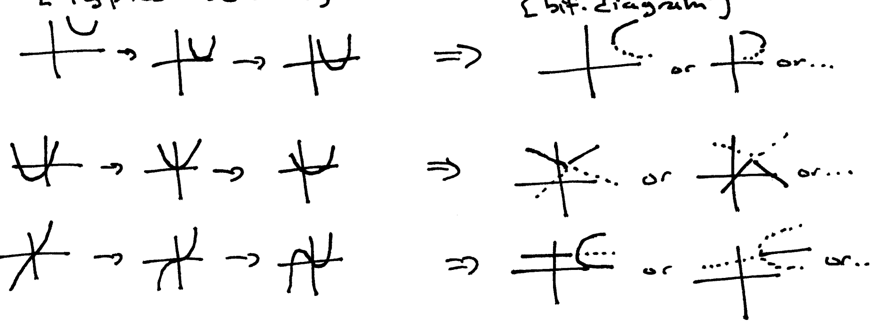
zeroes = fixed pts
>0 = increasing
<0 = decreasing

* linear stability analysis
  $\dot{x} = f(x)$, x_0 = fixed pt
  $f'(x_0) > 0 \Rightarrow$ ← • → $\Rightarrow$ unstable
  $f'(x_0) < 0 \Rightarrow$ → • ← $\Rightarrow$ stable
* Potential functions: $\dot{x} = f(x) = -\frac{dV}{dx}$
  $V(x)$ is potential, max $\Rightarrow$ ∩ $\Rightarrow$ unstable
  min $\Rightarrow$ ∪ $\Rightarrow$ stable

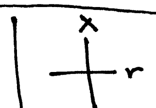


Bifurcations

* Bifurcation types [typical behavior] [bif. diagram]
  - Saddle node ⇒ → → ⇒ or or...
  - Transcritical ⇒ → → ⇒ or or...
  - Pitchfork ⇒ → → ⇒ or or...



Drawing Bifurcation Diagrams
(1) Find zeroes (i.e. $f(x) = 0$)
(2) Test stability in each segment
(3) Mark



* Changing variables, i.e., scale variables to eliminate constants
  [remember to use chain rule]

Circle

* Circle $\dot{\theta} = f(\theta)$ periodic solns ⟺ no fixed pts ⟺ $f(\theta) < 0$ or > 0 for all θ
  If periodic, Period = $\int_0^{2\pi} \frac{d\theta}{f(\theta)}$

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0

$a+d$ = trace = sum eigenval.
$ad-bc$ = det = product eigenval

Eigenvalue / Eigenvector

A a square matrix $Ax = \lambda x$ $x \neq 0$ is eigenvector, λ is eigenvalue

to find eigenvalues: $\det(A - \lambda I) = 0$, solve for λ

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightsquigarrow \lambda^2 - (a+d)\lambda + (ad-bc) = 0$

λ, μ eigenvalues for 2x2 matrix column of $A - \lambda I$ (if $\neq 0$) is eigenvector for μ

$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $e^{At} = I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots + A^k\frac{t^k}{k!} + \cdots$

* $Ae^{At} = e^{At}A$
* $e^{-At} = (e^{At})^{-1}$
* at $t=0$, $e^{At} = I$

if A not deficient (i.e. has full set of eigenvectors)

$\Rightarrow AS = SD$ or $A = SDS^{-1}$ so $e^{At} = S\underbrace{e^{Dt}}_{\text{easy since diagonal}}S^{-1}$

S = full set of eigenvectors put in columns
D = diagonal matrix of eigenvalues in same order as S

$\dot{x} = Ax + y$ then use "integrating factor"
$\dot{x} - Ax = y \rightsquigarrow (e^{-At}x)' = e^{-At}y$
$\rightsquigarrow e^{-At}x = C + \int e^{-At}y\,dt \Rightarrow x = e^{At}C + e^{At}\int e^{-At}y\,dt$