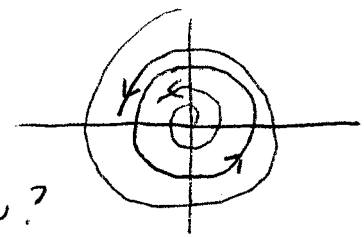
"I AM ALPHA AND OMEGA, THE FIRST AND THE LAST"
— REVELLATION 1:11

BASIC TOOL IN 2d TO SHOW EXISTENCE OF PERIODIC SOLUTION TO NONLINEAR SYSTEM IS POINCARÉ-BENDIXSON THEOREM (1901). BUT IT DOES MORE: DESCRIBES POSSIBLE LIMITS OF 2d TRAJECTORIES

7.3 AS $t \to \pm\infty$. e.g. $\vec{x}(t)$ MAY APPROACH A FIXED POINT, OR A LIMIT CYCLE AS IN

$$\dot{r} = r(1-r^2)$$
$$\dot{\theta} = 1$$



SOLUTIONS "APPROACH" $r=1$, $\theta = t$ AS $t \to \infty$. WHAT DOES THAT MEAN?
WHAT OTHER LIMITING BEHAVIOR IS POSSIBLE?

DEF: P IS AN ω - LIMIT POINT OF A TRAJECTORY $\vec{x}(t)$ IF $\lim_{n\to\infty} \vec{x}(t_n) = P$ FOR SOME SEQUENCE OF TIMES $t_n \to \infty$.
[α - LIMIT POINT IS SAME, FOR $t_n \to -\infty$]
$\omega(\vec{x})$ IS THE SET OF ALL ω - LIMIT POINTS OF THE TRAJECTORY STARTING AT POINT $\vec{x}$. SIM. $\alpha(\vec{x})$.

IN EXAMPLE, $\omega(x)$ = CIRCLE $r=1$ FOR ANY $\vec{x} \neq \vec{0}$.
$\alpha(x)$ = ORIGIN IF $\vec{x}$ INSIDE CIRCLE; CIRCLE ITSELF IF $\vec{x}$ ON CIRCLE; EMPTY IF $\vec{x}$ OUTSIDE CIRCLE.

PROP: AN ω LIMIT POINT IN A GRADIENT SYSTEM $\dot{x} = -\nabla V(x)$ CAN ONLY BE A FIXED POINT

PROOF: IF $\lim_{n\to\infty} x(t_n) = z$ THEN [CONTINUITY]

$\lim_{n\to\infty} V(x(t_n)) = V(z)$, SO $V(x(t))$ IS NONINCREASING, WITH LIMIT $V(z)$.
THEN $\dot{V} \to 0$ AS $t \to \infty$ AND [CONTINUITY] $\dot{V}(z) = 0$ TOO. SO $\nabla V(z) = 0$, A FIXED POINT.

DEF: A LIMIT CYCLE IS A CLOSED ORBIT C CONTAINED IN $\omega(x)$ [OR $\alpha(x)$] FOR SOME X NOT ON C.

QUESTION: WHAT CAN $\omega(x)$ BE, IN GENERAL?

PB THM (BOOK VERSION) SUPPOSE

1. R IS A COMPACT SUBSET OF $\mathbb{R}^2$
2. $\dot{x} = F(x)$ IS A SYSTEM WITH F C^1 ON AN OPEN SET CONTAINING R [GOES WITHOUT SAYING]
3. R CONTAINS NO FIXED POINTS
4. SOME TRAJECTORY C STARTS IN R AND STAYS IN R FOR ALL $t > 0$.

THEN, EITHER C IS A CLOSED ORBIT, OR IT APPROACHES A LIMIT CYCLE.

TO SHOW 4. WE USUALLY SHOW R IS A "TRAPPING REGION", OR [POSITIVELY] INVARIANT SET: EVERY TRAJECTORY WITH $x(0) \in R$ HAS $x(t) \in R$ FOR ALL $t > 0$.

PB THEOREM (FULL VERSION)

SUPPOSE Ω IS A NONEMPTY, COMPACT LIMIT SET OF A 2d SYSTEM, CONTAINING NO FIXED POINT. THEN Ω IS A CLOSED ORBIT.

CANONICAL EXAMPLE 7.3.1:
$$\dot{r} = r(1-r^2) + \mu r \cos\theta$$
$$\dot{\theta} = 1$$

FOR $\mu = 0$, WE SAW CLOSED ORBIT $r = 1$.
WHAT IF $0 < \mu < 1$?

SINCE $-1 \le \cos\theta \le 1$, ON A CIRCLE OF RADIUS r,

$$\dot{r} \le r(1-r^2) + \mu r = r(1+\mu - r^2)$$

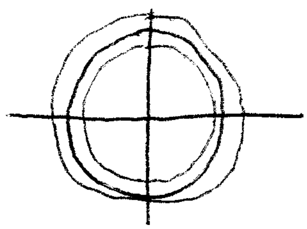
SO $\dot{r} < 0$ ON ANY CIRCLE WITH $r > \sqrt{1+\mu}$.

SIMILARLY, $\dot{r} \geq r(1-\mu-r^2)$, SO $\dot{r} > 0$ WHEN $r < \sqrt{1-\mu}$. SO, VELOCITY VECTOR FIELD POINTS <u>INTO</u> THE ANNULUS

$R = \{(r,\theta): r_{MIN} \leq r \leq r_{MAX}\}$, SAY WITH

$r_{MIN} = 0.999\sqrt{1-\mu}$, $r_{MAX} = 1.001\sqrt{1+\mu}$

SO R IS POS INVARIANT, CONTAINS NO FIXED POINT [ORIGIN] AND MUST CONTAIN A CLOSED ORBIT.



TO PROVE PB, WE NEED DEFINITIONS AND FACTS ABOUT LIMITS BEYOND THE USUAL CONTENT OF CALCULUS: ANALYSIS.

[DEFINITIONS, P. 10.1 FROM LAST YEAR]

USEFUL NOTATION: FOR $X \in \mathbb{R}^2$, $\phi_t(X)$ IS THE POINT $X(t)$ ALONG THE UNIQUE TRAJECTORY HAVING $X(0) = X$. THAT'S THE <u>SAME</u> AS $X(a+t)$ ON THE TRAJECTORY HAVING $X(a) = X$ FOR ANY a. "FLOW FOR TIME t" IS INDEP OF STARTING TIME.

DEFINITIONS [IN $\mathbb{R}^n$ FOR FIXED n]

X IS AN ACCUMULATION POINT OF A SEQUENCE $\{X_n\}$ IF EVERY NEIGHBORHOOD OF X CONTAINS X_n FOR INFINITELY MANY VALUES OF n.

X IS A LIMIT POINT OF A SET S IF EVERY NEIGHBORHOOD OF X CONTAINS A POINT (HENCE INFINITELY MANY) OF S OTHER THAN X ITSELF.

A SET S IS CLOSED IF IT CONTAINS ALL ITS LIMIT POINTS. EQUIVALENTLY, IF IT CONTAINS ALL ACCUMULATION POINTS OF ANY SEQUENCE OF ITS POINTS. OR, LIMIT OF ANY CONVERGENT SEQ OF POINTS

A SET IS COMPACT IF IT'S CLOSED AND BOUNDED (CONTAINED IN SOME BALL).

A SET IS OPEN IF IT CONTAINS A BALL AROUND EACH OF ITS POINTS ($\Leftrightarrow$ COMPLEMENT IS CLOSED)

$\lim_{t\to\infty} x(t) = Z$ MEANS FOR EACH NEIGHBORHOOD OF Z, THERE IS T SO THAT $x(t)$ IS IN THAT NBHD FOR ALL $t > T$. SAME FOR SEQUENCES.

BOLZANO-WEIERSTRASS THEOREM: ANY SEQUENCE OF POINTS IN A COMPACT SET HAS AN ACCUMULATION POINT. EQUIVALENTLY, A CONVERGENT SUBSEQUENCE. EQUIVALENTLY, ANY INFINITE SUBSET OF A COMPACT SET HAS A LIMIT POINT.

HAVING A SINGLE ACCUMULATION POINT $\neq$ CONVERGING.

PROPERTIES OF ω - LIMIT SETS:

1. IF X AND Z LIE ON THE SAME TRAJECTORY THEN $\omega(x) = \omega(z)$.
2. ANY ω - LIMIT SET IS CLOSED AND INVARIANT (BOTH POSITIVELY AND NEGATIVELY)
3. IF D IS A CLOSED, POSITIVELY INVARIANT SET AND $X \in D$ THEN $\omega(x) \subseteq D$.

PROOFS: (1) AS DEFINED, $\omega(x)$ IS A PROPERTY OF THE TRAJECTORY THROUGH X. SAME TRAJECTORY, SAME ω - LIMIT SET.

(2) TO SAY $\omega(X_0)$ IS CLOSED MEANS: IF $Y_1, Y_2, \ldots$ ARE IN $\omega(X_0)$ AND $Y_n \to Z$ AS $n \to \infty$, THEN Z IS ALSO IN $\omega(X_0)$. LET $x(t)$ BE THE TRAJECTORY STARTING AT X_0. FOR EACH n THERE IS t_n SO THAT $x(t_n)$ IS ARBITRARILY CLOSE TO Y_n. TO BE SPECIFIC LET'S DEMAND THAT

$$|x(t_n) - Y_n| < \frac{1}{n}.$$

THEN (TRIANGLE INEQUALITY)

$$|x(t_n) - Z| \leq |x(t_n) - Y_n| + |Y_n - Z|$$

$$< \frac{1}{n} + |Y_n - Z| \to 0 \quad \text{AS } n \to \infty$$

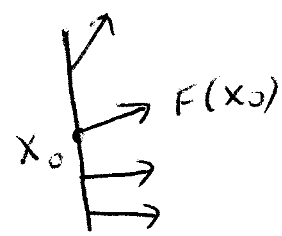
SO Z IS IN $\omega(X_0)$ BY DEFINITION.

TO SAY $\omega(X_0)$ IS INVARIANT MEANS: GIVEN ANY $Y \in \omega(X_0)$, $\phi_t(Y)$ IS IN $\omega(X_0)$ FOR EVERY t. SINCE $Y \in \omega(X_0)$, THERE EXIST t_n SUCH THAT $x(t_n) = \phi_{t_n}(X_0) \to Y$. THEN,

$\phi_{t+t_n}(x_0) = \phi_t(x(t_n)) \to \phi_t(Y)$ BY CONTINUITY OF THE FUNCTION ϕ_t. SO $\phi_t(Y) \in \omega(x_0)$.

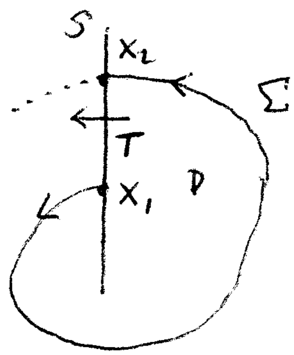
(3) SUPPOSE D IS CLOSED AND POSITIVELY INVARIANT, $x \in D$ AND $Y \in \omega(x)$. WE SHOW $Y \in D$. THERE EXIST t_n WITH $\phi_{t_n}(x) \to Y$. EACH $\phi_{t_n}(x)$ IS IN D [INVARIANT] SO THEIR LIMIT IS ALSO [CLOSURE].

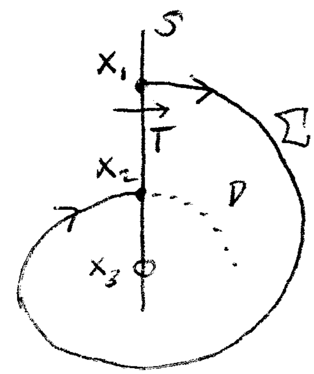
<u>DEF</u>: GIVEN A SYSTEM $\dot{x} = F(x)$ AND A POINT x_0 WHERE $F(x_0) \neq 0$, A <u>LOCAL</u> <u>SECTION</u> AT x_0 IS A LINE SEGMENT THROUGH x_0 <u>TRANSVERSE</u> [NOWHERE TANGENT] TO $F(x_0)$. IT ALWAYS EXISTS, BY CONTINUITY, AND $F(x)$ POINTS TOWARD THE <u>SAME</u> SIDE OF IT ALONG ITS LENGTH.



<u>LEMMA 1</u>: SUPPOSE S IS A LOCAL SECTION FOR $\dot{x} = F(x)$, AND A TRAJECTORY CROSSES IT AT SUCCESSIVE POINTS $x_1 = x(t_1)$, $x_2 = x(t_2)$, $x_3 = x(t_3)$ WITH $t_1 < t_2 < t_3$. THEN x_2 IS BETWEEN x_1 AND x_3 ALONG S. [GENERALIZES TO MORE THAN 3 POINTS]

<u>PROOF</u>: LET T BE THE SEGMENT OF S FROM x_1 TO x_2, AND Σ THE TRAJECTORY FROM x_1 TO x_2.





Σ AND T FORM A SIMPLE CLOSED CURVE BOUNDING A REGION D. [WE ASSUME THE JORDAN CURVE THEOREM: A SIMPLE CLOSED CURVE SEPARATES THE PLANE INTO INSIDE AND OUTSIDE REGIONS]

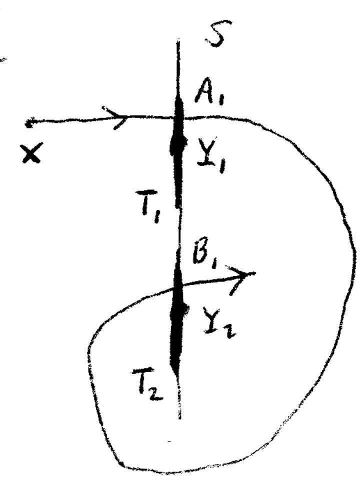
IF THE FLOW IS INTO D ALONG T, D IS POSITIVELY INVARIANT, THE TRAJECTORY BEYOND X_2 IS INSIDE D, AND X_3 MUST BE INSIDE D: BELOW X_2 ALONG S.

IF THE FLOW IS OUT OF D ALONG T, THE EXTERIOR OF D IS POSITIVELY INVARIANT, SO X_3 IS OUTSIDE D: ABOVE X_2 ALONG S.

INTUITIVELY: SUCCESSIVE CROSSINGS OF THE TRAJECTORY WITH S ARE ORDERED IN ONE OF THE TWO DIRECTIONS ALONG S; TRAJECTORIES SPIRAL OUTWARD OR INWARD BUT NOT BOTH.

LEMMA 2: IF $Y \in \omega(X)$ THEN THE TRAJECTORY THROUGH Y CROSSES ANY LOCAL SECTION AT ONE POINT AT MOST.

PROOF: SUPPOSE FOR CONTRADICTION THAT THE TRAJECTORY THROUGH Y CROSSES A LOCAL SECTION S AT TWO POINTS Y_1, Y_2. SINCE $\omega(X)$ CONTAINS Y AND IS INVARIANT, IT CONTAINS THE TRAJECTORY THROUGH Y, INCLUDING Y_1 AND Y_2. SO THE TRAJECTORY FROM X REPEATEDLY PASSES ARBITRARILY CLOSE TO BOTH Y_1 AND Y_2. LET T_1, T_2 BE DISJOINT SEGMENTS OF S CONTAINING



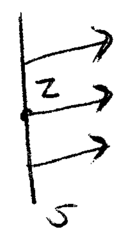
Y_1, Y_2 RESPECTIVELY. THEN THERE ARE SUCCESSIVE CROSSINGS $A_1, B_1, A_2, B_2, \ldots$ WITH $A_i \in T_1$, $B_i \in T_2$, OBVIOUSLY NOT ORDERED CORRECTLY ALONG S, VIOLATING LEMMA 1.

PROOF OF POINCARE-BENDIXSON THEOREM :

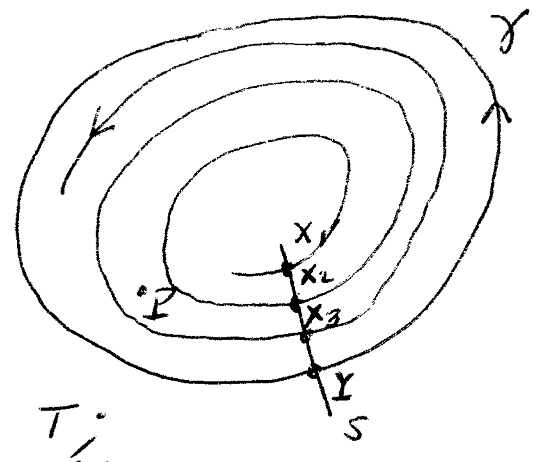
[SUPPOSE Ω IS A NONEMPTY, COMPACT LIMIT SET OF A 2d SYSTEM, CONTAINING NO FIXED POINT. THEN Ω IS A SINGLE CLOSED ORBIT.]

LET $\Omega = \omega(X)$ AND PICK $Y \in \Omega$. WE SHOW THAT $\phi_t(Y)$, THE TRAJECTORY THROUGH Y, IS A CLOSED ORBIT. Ω IS INVARIANT, SO IT CONTAINS $\phi_t(Y)$. IT IS CLOSED AND BOUNDED, SO IT CONTAINS $\omega(Y)$ WHICH IS NONEMPTY BY BOLZANO-WEIERSTRASS. CHOOSE $Z \in \omega(Y)$ AND A LOCAL SECTION S AT Z [WHICH IS NOT A FIXED POINT!]. BY LEMMA 2, $\phi_t(Y)$ CROSSES S AT ONE POINT ONLY.

BUT $\phi_t(Y)$ MUST REPEATEDLY COME ARBITRARILY CLOSE TO Z, WHERE THE FLOW IS ACROSS S, SO IT CROSSES S AT AN INFINITE SEQUENCE OF TIMES. THE ONLY POSSIBILITY IS THAT ALL THESE CROSSINGS OCCUR AT Z, SO $\phi_t(Y)$ IS A PERIODIC ORBIT THROUGH Z.



NOW WE KNOW $\Omega = \omega(X)$ CONTAINS A CLOSED ORBIT γ. WE MUST FINALLY SHOW $\Omega = \gamma$. γ IS THE TRAJECTORY THROUGH Y, AND $Y \in \omega(X)$, SO $\phi_t(X)$ REPEATEDLY COMES ARBITRARILY CLOSE TO Y. TAKE A LOCAL SECTION S AT Y; $\phi_t(X)$ WILL CROSS IT AT SUCCESSIVE ORDERED POINTS $X_1, X_2, X_3, \ldots$



IF T IS THE PERIOD OF γ, ONCE X_n IS SUFFICIENTLY CLOSE TO Y, CONTINUITY OF THE FLOW WITH RESPECT TO INITIAL CONDITIONS GUARANTEES THAT $\phi_t(x)$ WILL STAY CLOSE TO γ FOR THE TIME [APPROXIMATELY T; SURELY LESS THAN $2T$] REQUIRED TO RETURN TO X_{n+1}, WHICH IS EVEN CLOSER TO γ. THUS $\phi_t(x)$ GETS ARBITRARILY CLOSE TO γ FOR t LARGE ENOUGH.

IF P IS <u>NOT</u> ON γ, $\phi_t(x)$ CANNOT RETURN ARBITRARILY CLOSE TO P, SO P CANNOT BE IN $\omega(x)$. THUS $\omega(x)$ CONSISTS OF γ ONLY.

FINALLY, DEDUCE TEXTBOOK'S VERSION: IF A TRAJECTORY STARTS IN AND REMAINS IN A COMPACT, REGION R CONTAINING NO FIXED POINTS, IT EITHER <u>IS</u> OR <u>APPROACHES</u> A CLOSED ORBIT.

<u>PROOF</u>: SUCH A TRAJECTORY HAS A NONEMPTY ω-LIMIT SET [BOLZANO-WEIERSTRASS] Ω, WHICH IS A CLOSED ORBIT γ BY PB THEOREM. PROOF OF PB SHOWS TRAJECTORY EITHER IS, OR APPROACHES, γ.