Math 130B – Prof. Rabin – Final Examination – March 18, 2008

This take-home final exam is due at Prof. Rabin's office (APM 6220) or the homework drop box, by noon on Friday, March 21, 2008. You may use your textbook, class notes, and homework, but no other assistance. Show all work, and explain all reasoning. If any problem is not clear, you may email me a question. Each of the seven problems is worth 25 points. Good luck!

(1) Show that the system

$$\dot{x} = x(y-1), \qquad \dot{y} = x + y - 2y^2$$

has no closed orbits.

(2) A particle moves along the x axis according to the equation (Newton's Law of Motion)

$$\ddot{x} = x^4 - x^2.$$

(a) Rewrite the equation as a linear system by introducing $y = \dot{x}$. Find and classify the fixed points.

(b) Find the energy function $E(x, y)$ for this conservative system. For what range of energy values will there be bounded (closed) orbits?

(c) For a particular orbit whose energy is in the range you determined above, with $x(0) = x_0$ and $y(0) = 0$, set up an integral giving the period of the orbit. Do not try to evaluate it. Your integral may include symbols for quantities you cannot compute explicitly, provided you define them.

(3) Consider the relaxation oscillator in Lienard form,

$$\dot{x} = \mu[y - F(x)], \qquad \dot{y} = -x/\mu.$$

In this problem, $F(x)$ is given by $x(x-2)$ when $x \geq 0$, and by $-x(x+2)$ when $x \leq 0$.

Sketch the limit cycle in the phase plane in the limit of very large μ, and compute its approximate period.

(4) This problem concerns a predator-prey system given by

$$\dot{x} = x[x(1-x) - y], \qquad \dot{y} = y(x-a).$$

Here a is a parameter in the range $0 < a < 1$, and $x, y \geq 0$.

(a) Show that there is a bifurcation in this system when $a = 1/2$.

(b) By considering $\frac{d}{dt}(x+y)$, show that the region bounded by the coordinate axes and the line $x + y = C$ is a trapping region for large enough C.

(c) Show that a closed orbit exists when $a < 1/2$. The bifurcation is therefore a Hopf bifurcation.

MORE PROBLEMS ON NEXT PAGE!

(5) This problem concerns the system

$$\dot{x} = (1 - x^2)(x + 2y), \qquad \dot{y} = (1 - y^2)(-2x + y).$$

Let S be the square with vertices at the four points $(\pm 1, \pm 1)$, and let R be the region inside it, that is, $R = \{(x, y) : \ -1 < x < 1, \ -1 < y < 1\}$. We are interested in trajectories that start within R.

(a) Show that the origin and the vertices of S are fixed points. Show that the vertices are saddle points while the origin is a source.

(b) Show that the function $V(x, y) = (1 - x^2)(1 - y^2)$ decreases along every nontrivial trajectory in R: $dV/dt < 0$ for $(x, y) \neq (0, 0)$.

(c) Show that S is an invariant set.

(d) Give as careful a proof as you can that, for every $\mathbf{x}$ in R other than the origin, $\omega(\mathbf{x}) = S$. (Thus, S is an example of a cycle graph.)

(6) The second-order equation

$$\ddot{x} + \epsilon \dot{x}^2 + x = 0$$

describes a harmonic oscillator with a small ($\epsilon << 1$) additional nonlinear force proportional to the velocity squared. Use perturbation theory to find the solution satisfying the initial conditions $x(0) = 1, \ \dot{x}(0) = 0$. Specifically, assume a power series solution of the form $x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots$.

(a) Find the differential equations satisfied by the first three functions x_0, x_1, and x_2.

(b) Solve for x_0 and x_1, and show that to this level of approximation the oscillator actually oscillates around the point $x = -\epsilon/2$ rather than $x = 0$. Make sure your solution satisfies the initial conditions. You may need the identity $\sin^2 t = (1 - \cos 2t)/2$.

(c) You do *not* need to solve for x_2, but it turns out that it contains the secular term $(1/6) t \sin t$. Explain how this is interpreted as a change in the frequency of the oscillator. What is the frequency, to this level of approximation?

(7) Consider the following three-dimensional system:

$$\dot{x} = y, \qquad \dot{y} = x(1 - z) - y, \qquad \dot{z} = -z + x^2.$$

Find the fixed points, and classify them by linearization if possible. You may encounter a cubic equation for some eigenvalues, but at least one root should be easy to guess.