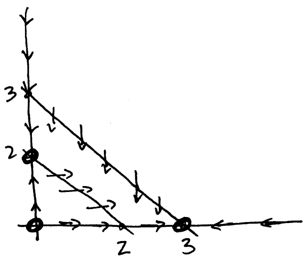
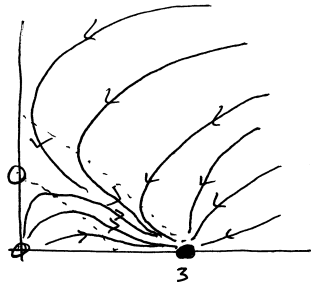
Homework 1 solutions

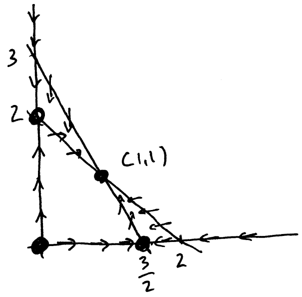
6.4.1 The fixed points will occur when $\dot{x} = 0$ and $\dot{y} = 0$. So at the fixed point we have that [either $x = 0$ or $x + y = 3$] and [either $y = 0$ or $x + y = 2$]. This leads to four possible combinations $x = 0$, $y = 0$ gives $(x^*, y^*) = (0, 0)$; $x = 0$, $x + y = 2$ gives $(x^*, y^*) = (0, 2)$; $2x + y = 3$, $y = 0$ gives $(x^*, y^*) = (3/2, 0)$; $x + y = 3$, $x + y = 2$ gives no solutions. So there are only three fixed points. The nullclines are given when $\dot{x} = 0$ (so pass through vertically) and $\dot{y} = 0$ (so pass through horizontally). A sketch of the nullclines is shown below on the left and a "plausible" phase portrait is shown below on the right. In particular the whole first quadrant is a basin of attraction to the point $(3, 0)$.

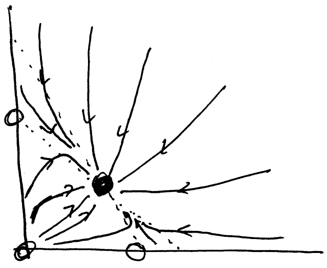




6.4.2 The fixed points will occur when $\dot{x} = 0$ and $\dot{y} = 0$. So at the fixed point we have that [either $x = 0$ or $2x + y = 3$] and [either $y = 0$ or $x + y = 2$]. This leads to four possible combinations $x = 0$, $y = 0$ gives $(x^*, y^*) = (0, 0)$; $x = 0$, $2x + y = 2$ gives $(x^*, y^*) = (0, 1)$; $x + y = 3$, $y = 0$ gives $(x^*, y^*) = (3, 0)$; $2x + y = 3$, $x + y = 2$ gives $(x^*, y^*) = (1, 1)$. So there are four fixed points. The nullclines are given when $\dot{x} = 0$ (so pass through vertically) and $\dot{y} = 0$ (so pass through horizontally). A sketch of the nullclines is shown below on the left and a "plausible" phase portrait is shown below on the right.

In particular the whole first quadrant is a basin of attraction to the point $(1, 1)$.





6.5.3 Rearranging and multiplying by $\dot{x}$ we have

$$\ddot{x}\dot{x} - (a - e^x)\dot{x} = 0.$$

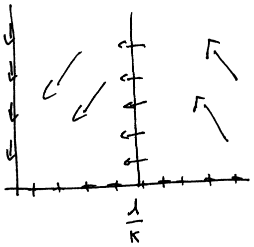
Integrating both sides, using the substitution rule, we have

$$\underbrace{\frac{1}{2}(\dot{x})^2 - ax + e^x}_{=\text{conserved quantity}} = C.$$

To sketch the phase portrait we can sketch level curves of the conserved quantity.

6.5.6 (a) The fixed points occur when $\dot{x} = -kxy = 0$ and $\dot{y} = kxy - \ell y = 0$. It can be checked that the fixed points are along the line $y = 0$. We will distinguish one point on that line, namely $(x^*, y^*) = (\ell/k, 0)$ as a possibly important point since this corresponds to the "non-trivial" part of $\dot{y} = 0$.

(b) A sketch of the nullclines and vector field are shown below.



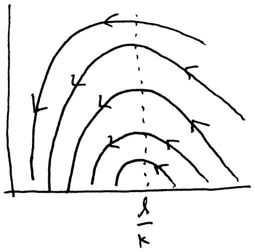
(c) We have that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{kxy - \ell y}{-kxy} = -1 + \frac{\ell}{k}\frac{1}{x}.$$

Integrating it follows that

$$y = -x + \frac{\ell}{k}\ln(x) + C \qquad \text{or} \qquad \underbrace{y + x - \frac{\ell}{k}\ln(x)}_{=\text{conserved quantity}} = C$$

(d) Using the sketch from part (b) we get the following phase portrait. Note in particular that given any starting point that as $t \to \infty$ that $y \to 0$ (though it might first initially increase). Intuitively this makes sense since y is the number of sick people so eventually the epidemic will die out.



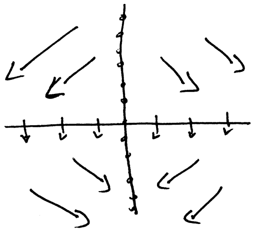
(e) In order to have an epidemic we need $\dot{y} > 0$ which since k and y are positive translates into the condition that $x > \ell/k$. Note that this agrees with the picture given in part (d).

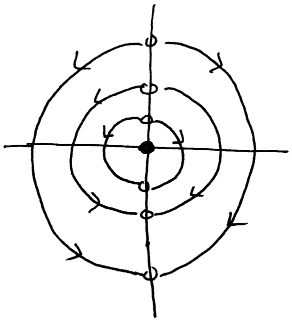
6.5.12 (a) To show that the energy is conserved we show that it is constant, i.e., that the derivative is 0. So we have that

$$\frac{d}{dt}(E) = 2x\dot{x} + 2y\dot{y} = 2x(xy) + 2y(-x^2) = 0.$$

(b) Since at $(0,0)$ both $\dot{x} = 0$ and $\dot{y} = 0$ it follows that the origin is a fixed point. But more generally since x shows up in both $\dot{x}$ and $\dot{y}$ then the entire line of points on $y = 0$ are fixed points. Since the origin is on that line it is not an isolated fixed point.

(c) To see what does happen let us sketch the nullclines and the phase portrait. In particular this will give us the pictures shown below.





6.5.13 (a) Turning this into a system by $\dot{x} = y$ and $\dot{y} = \ddot{x} = -x - \varepsilon x^3$, we see that the fixed points will happen when $y = 0$ and $x(1 + \varepsilon x^2) = 0$. For the case when $\varepsilon > 0$ this gives a unique fixed point at the origin. The trajectories of this system correspond to the level curves of the "energy" which in terms of x and y can be shown to be $\frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{\varepsilon}{4}x^4 = C$ as C (≥ 0) varies. These curves are somewhat similar to circles and will fill up the space and so the origin is a nonlinear center.

(b) Starting as before this turns into the system $\dot{x} = y$ and $\dot{y} = -x - \varepsilon x^3$. Now however there will be 3 fixed points corresponding to $y = 0$ and $x = -\sqrt{-1/\varepsilon}$, 0, $\sqrt{-1/\varepsilon}$. So the origin *cannot* be a center for the whole space (i.e., the other fixed points do not lie in a closed circuit that goes around the origin). So now we apply Theorem 6.5.1. First we know that the origin is an isolated fixed point. So it remains to show that $(0,0)$ is a *minimum* of the conserved quantity $E(x,y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{\varepsilon}{4}x^4$. This involved some simple calculus, namely we need to show that $E_x(0,0) = 0$, $E_y(0,0) = 0$, $E_{xx}(0,0)E_{yy}(0,0) - (E_{xy}(0,0))^2 > 0$ and $E_{xx}(0,0) > 0$. All of these are easily verified. It now follows that locally all the trajectories *close* to the origin are closed.

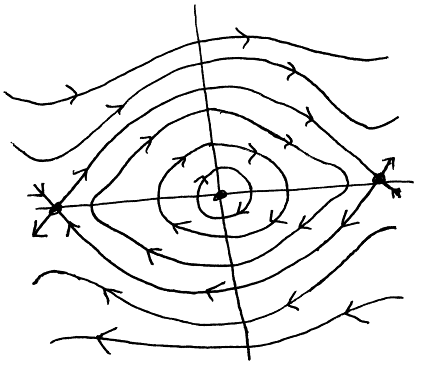
6.5.15 (a) If we let $\tau = \omega t$ then we have that $\ddot{\phi} = \omega^2\phi''$. Making this change of variables we have that

$$mr\phi'' = -mg\sin\phi + mr\omega^2\sin\phi\cos\phi \qquad \text{becomes} \qquad mr\omega^2\phi'' = -mg\sin\phi + mr\omega^2\sin\phi\cos\phi.$$

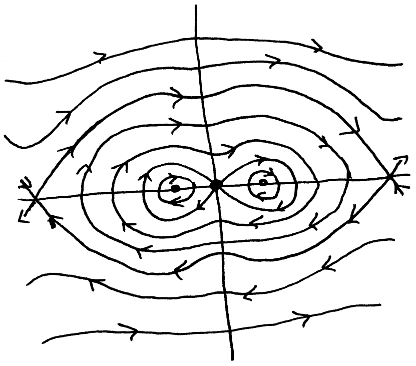
If we now divide both sides by $mr\omega^2$ and then factor the right hand side we have

$$\phi'' = \sin\phi\big(\cos\phi - \underbrace{\frac{g}{r\omega^2}}_{=\gamma^{-1}}\big).$$

(b) The different phase portraits correspond to when $0 < \gamma < 1$ (i.e., 0 is the only equilibrium) and when $\gamma > 1$ when there are two additional equilibria. If we let $x = \phi$ and $y = \phi'$ then it is easy to see that the trajectories of the phase portrait satisfy $\frac{1}{2}y^2 + \frac{1}{4}\cos(2x) - \gamma^{-1}\cos(x) = C$. Using a computer algebra system it is then easy to draw the level curves and get the two following phase portraits.



$0<\gamma<1$



$1<\gamma$

(c) Looking at the phase portraits we see that almost always we will be either continually circling the hoop in one direction or oscillating around an equilibrium point. When $0 < \gamma < 1$ there is only the single equilibrium at $\phi = 0$ that we could oscillate around while if $1 < \gamma$ then there are three different equilibria, either a huge oscillation around 0 or small oscillations at points near 0.

6.5.16 We start (replacing θ by x to help out intuition and drop the term with b) and get $\ddot{x} = -\frac{g}{r}\sin x\big(1 - \frac{r\omega^2}{g}\cos x\big)$. If we now turn this into a system by $\dot{x} = y$ and $\dot{y} = \ddot{x} = -\frac{g}{r}\sin x\big(1 - \frac{r\omega^2}{g}\cos x\big)$. For a fixed point we need to have $y = 0$ and either $x = 0$ (already known) or $\cos x = \frac{g}{r\omega^2}$, the latter case will produce a symmetric pair of fixed points as long as $\frac{g}{r\omega^2} < 1$, i.e., as long as $\omega > \sqrt{g/r}$ (in other words as long as ω is sufficiently large). If we now look at the linearization around these points we have that around (either) of the fixed points it locally linearizes to

$$\begin{aligned}\dot{x} &= y;\\ \dot{y} &= \frac{g^2 - r^2\omega^4}{r^2\omega^2}.\end{aligned}$$

The calculation for $\dot{x}$ relied on knowing that at the fixed point $\cos x = \frac{g}{r\omega^2}$ and also that $\sin^2 x = 1 - \cos^2 x$.

If we compute the eigenvalues of this system we get the complex numbers

$$\lambda = \pm i\frac{\sqrt{r^2\omega^4 - g^2}}{r\omega}.$$

In particular the frequency of the linear system, which is an approximation for the frequency of small oscillations around the equilibria are given by the complex part of the eigenvalues. Therefore the frequency is approximately $\sqrt{r^2\omega^4 - g^2}/r\omega$.

6.5.19 (a) For both rabbits and foxes there are two forces, namely internal (i.e., among themselves) and external (i.e., between the two species).

For rabbits they tend to breed like crazy and so the term aR represents that the rabbit population would grow in the absence of foxes. The interaction between rabbits and foxes is bad for the rabbits and the greater the number of foxes/rabbits the more negative the effect on the population of rabbits, this explains the term $-bRF$.

For foxes they might be loners and uncooperative and so might fight amongst themselves leading to a decreasing population in the absence of rabbits explaining the $-cF$ term. On the other hand a large number of foxes/rabbits tends to mean more food and so has a positive impact on population explaining the term dRF.

Of course this oversimplifies things, for instance the rabbit population cannot get arbitrarily large but is self limited (i.e., recall the logisitic model from last quarter).

(b) We rescale as follows $R = \alpha x$, $F = \beta y$ and $t = \gamma\tau$. Doing this our system of equations then becomes

$$\begin{aligned}\tfrac{\alpha}{\gamma}x' &= a\alpha x - b\alpha\beta xy\\ \tfrac{\beta}{\gamma}y' &= -c\beta y + d\alpha\beta xy\end{aligned} \qquad \text{or} \qquad \begin{aligned}x' &= a\gamma x\big(1 - \tfrac{b\beta}{a}y\big)\\ y' &= c\gamma y\big(\tfrac{d\alpha}{c}x - 1\big)\end{aligned}.$$

To get the desired form we want $a\gamma = 1$, $b\beta/a = 1$ and $d\alpha/c = 1$, i.e., we need $\gamma = 1/a$, $\beta = a/b$ and $\alpha = c/d$. So if we let $\mu = c/a$ then we can conclude that in dimensionless form

$$\begin{aligned}x' &= x(1-y)\\ y' &= \mu y(x-1)\end{aligned}.$$

(c) In the dimensionless coordinate system we repeat an earlier trick and find dy/dx and then solve. In particular, we have that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\mu y(x-1)}{x(1-y)},$$

which rearranges to

$$\big(\frac{1}{y} - 1\big)\, dy = \mu\big(1 - \frac{1}{x}\big)\, dx.$$

Now integrating we have

$$\ln(y) - y + C = \mu\big(x - \ln(x)\big),$$

which rearranges to

$$C = \underbrace{\mu\big(x - \ln(x)\big) + \big(y - \ln(y)\big)}_{=\text{conserved quantity}}.$$

(d) The trajectories are the level curves of the conserved quantity that we found in part (c). It can be shown that the level curves of this function is the single point at $(1,1)$ for $C = \mu + 1$ and that for $C > \mu + 1$ the level curves are closed loop [one can use computers to graph some of these curves for intuition, if needed]. In particular we end up in a cycle for any nonzero starting point away from $(1,1)$.