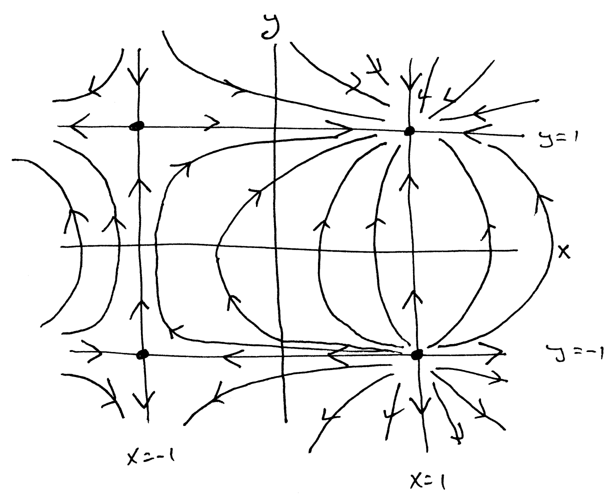
Homework 2 solutions

6.6.1 To check if the system is reversible we replace y and $\dot{x}$ by $-y$ and $-\dot{x}$ respectively (i.e., replace t by $-t$ and y by $-y$, note that $\dot{y}$ thus becomes $(-)(-)\dot{y} = \dot{y}$); if it reduces to what we started with then it is reversible. For our system this gives

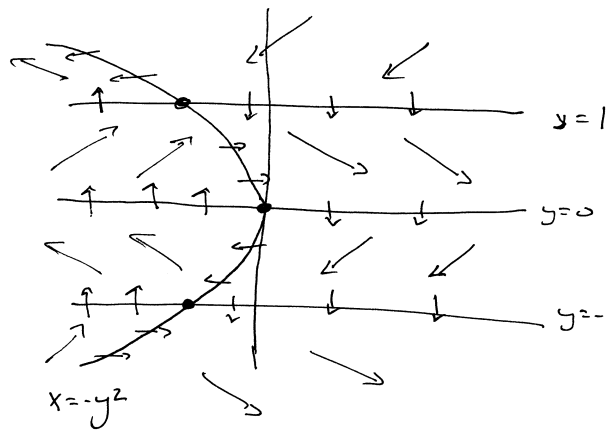
$$\begin{aligned} -\dot{x} &= -y(1-x^2) \\ \dot{y} &= 1-(-y)^2 \end{aligned} \qquad \text{simplifying to} \qquad \begin{aligned} \dot{x} &= y(1-x^2) \\ \dot{y} &= 1-y^2 \end{aligned} .$$

This shows that the system is reversible.

To plot the function we can sketch the nullclines and then get a rough sketch of the system as shown below. Alternatively we can see that the phase portrait will be the level curves of $F(x, y) = (1+x)(1-y^2)/(1-x)$ (i.e., we have $dy/dx = (1-y^2)/y(1-x^2)$ so rearrange and use partial fractions to solve).



6.6.6 a) and b) The nullclines are $y = -1, 0, 1$ (for which we pass through vertically) and $x = -y^2$ (for which we pass through horizontally). These nullclines and the directions in the different regions of the plane are shown below.



c) The linearization at $(-1, \pm 1)$ is given by

$$\left.\begin{bmatrix} 0 & 1-3y^2 \\ -1 & -2y \end{bmatrix}\right|_{(x,y)=(-1,\pm 1)}.$$

So at $(-1, 1)$ this becomes

$$\begin{bmatrix} 0 & -2 \\ -1 & -2 \end{bmatrix}.$$

The characteristic equation gives $\lambda^2 + 2\lambda - 2 = 0$ giving eigenvalues of $-1+\sqrt{3}$ and $-1-\sqrt{3}$ which respectively have $\left[\begin{smallmatrix} 1+\sqrt{3} \\ -1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1-\sqrt{3} \\ -1 \end{smallmatrix}\right]$ as eigenvectors.
At $(-1, -1)$ this becomes

$$\begin{bmatrix} 0 & -2 \\ -1 & 2 \end{bmatrix}.$$

The characteristic equation gives $\lambda^2 - 2\lambda - 2 = 0$ giving eigenvalues of $1+\sqrt{3}$ and $1-\sqrt{3}$ which respectively have $\left[\begin{smallmatrix} -1+\sqrt{3} \\ -1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} -1-\sqrt{3} \\ -1 \end{smallmatrix}\right]$ as eigenvectors.

d) Looking at the answer to part (b) we see that the sign of $\dot{y} > 0$ and the sign of $\dot{x} < 0$ so that it will continue moving up and to the left, eventually it must hit the axis. By reversibility the mirror image of the path between $(-1, -1)$ and the negative x-axis will go to $(-1, 1)$ giving us a trajectory connecting $(-1, -1)$ and $(-1, 1)$.

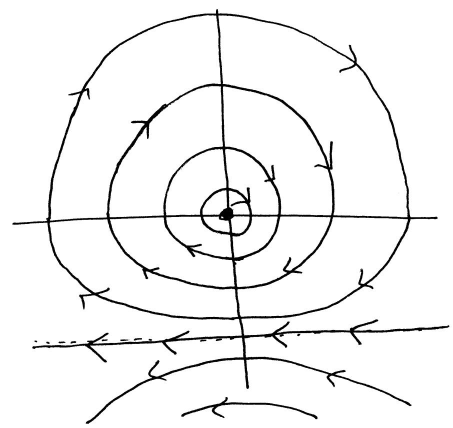
e) Similarly if we start at $(-1, 1)$ and move out down and to the left then $\dot{y} < 0$ and $\dot{x} > 0$ so that eventually we will hit the positive x-axis. Again by reversibility the mirror image of the path between $(-1, 1)$ and the positive x-axis will go to $(-1, -1)$ giving us the second trajectory connecting $(-1, 1)$ and $(-1, -1)$. It is then easy to fill in remaining trajectories to get Figure 6.6.4 in the book.

6.6.7 Starting with $\ddot{x} + x\dot{x} + x = 0$ we can translate this into the system $\dot{x} = y$ and $\dot{y} = \ddot{x} = -x\dot{x} - x = -xy - x$. If we let t be replaced by $-t$ and x by $-x$ (here using a different form of reversibility) then we see that

$$\begin{aligned} \dot{x} &= y \\ -\dot{y} &= -(-x)y - (-x) \end{aligned} \qquad \text{simplifying to} \qquad \begin{aligned} \dot{x} &= y \\ \dot{y} &= -xy - x \end{aligned} .$$

This shows that the system is reversible.

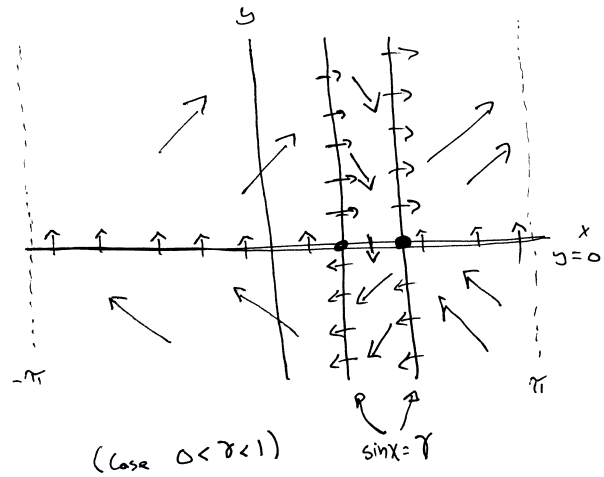
A rough sketch of the phase portrait is shown below [found using nullclines and intuition, alternatively you can show that a conserved quantity is $y - \ln|y+1| + \frac{1}{2}x^2$ and use this to draw level curves].



6.7.2 a) If we translate this into a system we have $\dot{x} = y$ and $\dot{y} = \ddot{x} = \gamma - \sin x$. The equilibrium points then occur when $y = 0$ and $\sin x = \gamma$. This gives two equilibrium points when $-1 < \gamma < 1$, one equilibrium point when $\gamma = \pm 1$ and 0 equilibrium points otherwise. Note that the linearization around these points is given by the matrix

$$\left.\begin{bmatrix} 0 & 1 \\ -\cos x & 0 \end{bmatrix}\right|_{\sin x = \gamma} .$$

In particular the eigenvalues are $\pm\sqrt{-\cos x}$. In the range $-1 < \gamma < 1$ this will give one equilibrium point which is a saddle (i.e., two distinct real eigenvalues when $\cos x < 1$) and one equilibrium point which is a center (i.e., pure imaginary eigenvalues when $\cos x > 1$). When $\gamma = \pm 1$ then the nodes become degenerate.

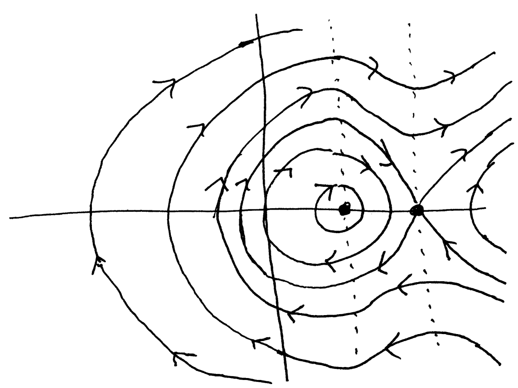
b) 

c) The system is conservative, i.e., rearranging we have that $\ddot{x} + \sin x - \gamma = 0$. Now multiplying by $\dot{x}$ and integrating we get $\frac{1}{2}y^2 - \cos x - \gamma x = C$, so that the conserved quantity is $\frac{1}{2}y^2 - \cos x - \gamma x$.

The system is also reversible. If we replace t by $-t$ and y by $-y$ then we have

$$\begin{aligned} -\dot{x} &= -y \\ (-)(-)\dot{y} &= \gamma - \sin x \end{aligned} \qquad \text{simplifying to} \qquad \begin{aligned} \dot{x} &= y \\ \dot{y} &= \gamma - \sin x \end{aligned},$$

showing that the system is reversible.

d) 

e) Similar to the problem from last week the frequency will be approximately the frequency of the linearization, which is $\sqrt{\cos x}$ when $\sin x = \gamma$. This simplifies as

$$\sqrt{\cos x} = \sqrt[4]{\cos^2 x} = \sqrt[4]{1 - \sin^2 x} = \sqrt[4]{1 - \gamma^2}.$$

6.7.4 a) Starting with $\ddot{\theta} + \sin\theta = 0$ we multiply by $\dot{\theta}$ and integrate to get $\frac{1}{2}(\dot{\theta})^2 - \cos\theta = C$. At the maximum amplitude α we know that $\dot{\theta} = 0$ implying that $C = -\cos\alpha$.

Rearranging we can conclude that $\dot{\theta}^2 = 2(\cos\theta - \cos\alpha)$. The period of the pendulum will be four times the amount of time it takes to swing between 0 and α so that the period will be

$$T = 4\int_0^\alpha \frac{d\theta}{[2(\cos\theta - \cos\alpha)]^{1/2}}.$$

b) Note that $\cos(2x) = 1 - 2\sin^2(x)$, or replacing x by $\theta/2$ and $\alpha/2$ we have that

$$2(\cos\theta - \cos\alpha) = 2\big((1 - 2\sin^2\frac{\theta}{2}) - (1 - 2\sin^2\frac{\alpha}{2})\big) = 4(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}).$$

Substituting this into T we now have

$$T = 4\int_0^\alpha \frac{d\theta}{[4(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2})]^{1/2}}.$$

c) We introduce (as directed) a new variable ϕ by $\sin\frac{\alpha}{2}\sin\phi = \sin\frac{\theta}{2}$. For our bounds note that at $\theta = 0$ we need $\sin\phi = 0$ so that we can use $\phi = 0$ and that at $\theta = \alpha$ we need $\sin\frac{\alpha}{2}\sin\phi = \sin\frac{\alpha}{2}$ so that we need $\sin\phi = 1$ and so we can use $\phi = \frac{\pi}{2}$. Also note that we have that $\sin\frac{\alpha}{2}\cos\phi\, d\phi = \frac{1}{2}\cos\frac{\theta}{2}\, d\theta$ or $d\theta = 2\sin\frac{\alpha}{2}\cos\phi\, d\phi / \cos\frac{\theta}{2}$.

Making substitutions to our integral this becomes

$$T = 4\int_0^{\pi/2} \frac{2\sin\frac{\alpha}{2}\cos\phi\, d\phi/\cos\frac{\theta}{2}}{[4\sin^2\frac{\alpha}{2}\underbrace{(1 - \sin^2\phi)}_{=\cos^2\phi}]^{1/2}} = 4\int_0^{\pi/2}\frac{d\phi}{\cos\frac{\theta}{2}}.$$

This last integral can be rewritten as follows

$$\begin{aligned} 4\int_0^{\pi/2}\frac{d\phi}{\cos\frac{\theta}{2}} = 4\int_0^{\pi/2}\frac{d\phi}{[\cos^2\frac{\theta}{2}]^{1/2}} &= 4\int_0^{\pi/2}\frac{d\phi}{[1 - \sin^2\frac{\theta}{2}]^{1/2}} \\ &= 4\int_0^{\pi/2}\frac{d\phi}{[1 - \sin^2\frac{\alpha}{2}\sin\phi]^{1/2}} = 4K(\sin^2\frac{\alpha}{2}), \end{aligned}$$

where $K(m)$ is an elliptic integral as described in the book.

d) First note that $\sin^k\frac{\alpha}{2} = \frac{\alpha^k}{2^k} + O(\alpha^{k+2})$, so any term that we will encounter which has a $\sin^4\frac{\alpha}{2}$ (or higher power) is a lower order term and we can put it aside. We also note that by the binomial theorem that $(1-x)^{-1/2} = 1 + \frac{1}{2}x + O(x^2)$. From this we have that

$$
\begin{aligned}
4K(\sin^2 \frac{\alpha}{2}) &= 4\int_0^{\pi/2} \frac{d\phi}{(1-\sin^2\frac{\alpha}{2}\sin^2\phi)^{1/2}} \\
&= 4\int_0^{\pi/2} (1+\frac{1}{2}\sin^2\frac{\alpha}{2}\sin^2\phi)\,d\phi + O(\alpha^4) \\
&= 4[\frac{\pi}{2} + \frac{1}{2}\sin^2\frac{\alpha}{2}\frac{\pi}{4} + O(\alpha^4)] \\
&= 4[\frac{\pi}{2} + \frac{1}{32}\alpha^2\pi + O(\alpha^4)] \\
&= 2\pi[1 + \frac{1}{16}\alpha^2 + O(\alpha^4)].
\end{aligned}
$$

In going from the second to third line we used that $\int_0^{\pi/2} \sin^2\phi\,d\phi = \pi/4$ and in going from the third to the fourth line we used the fact about expansion noted above.