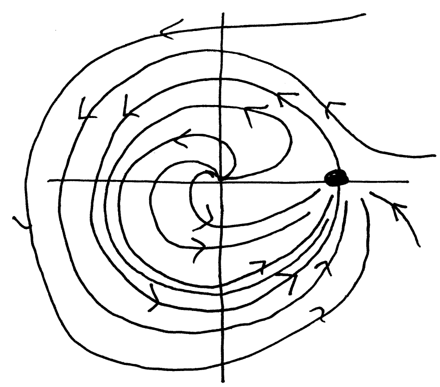
Homework 3 solutions

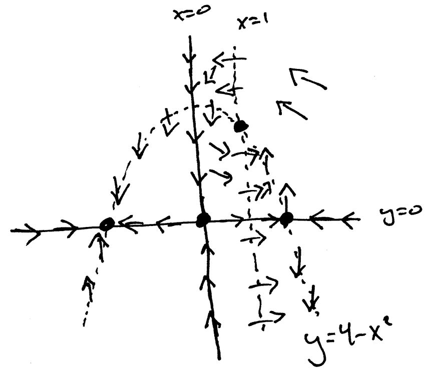
6.3.15 We have that $\dot{r} = r(1 - r^2)$ so that r is increasing for $0 < r < 1$ and decreasing for $1 < r$ so that except for the origin we move to $r = 1$. Similarly for $\dot{\theta} = 1 - \cos\theta$ we have that for $\theta \neq 0$ that $\dot{\theta} > 0$ (i.e., the angle moves counterclockwise and heads to $\theta = 0$). Thus every point other than the origin will, in the long run, move to $r = 1$ and $\theta = 0$. In particular this is attracting. However it is not Liapunov stable since we can start very close to the fixed point with $\theta > 0$, but the trajectory will have to go "all the way around" the circle, i.e., does not stay close to the fixed point.

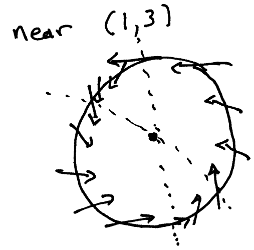
A sketch of the phase portrait is shown below.



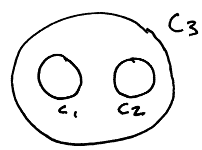
6.8.2 The only fixed point for $\dot{x} = x^2$ and $\dot{y} = y$ clearly occurs at $(x^*, y^*) = (0, 0)$. For a small circle drawn around zero we see that the flow of the vector field never goes to the left. In particular when we circle around the origin once we will not have made a rotation and so the origin has index 0.

6.8.7 This is very similar to Example 6.8.5 and we can try the same approach. First off notice that the nullclines for when we cross vertically are when $x = 0$ and $y = 4 - x^2$ and the nullclines for when we pass horizontally are $y = 0$ and $x = 1$. We first observe that the possible fixed points are at $(0, 0)$, $(1, 3)$, $(2, 0)$ and $(-2, 0)$. Next thing to observe is that on the axes we know the trajectories, and in particular no closed orbit can intersect an axis. Since a closed orbit must enclose a fixed point that leaves us with the only possibility of having an orbit in the first quadrant enclosing the point $(1, 3)$. Examining a small closed loop around $(1, 3)$ however we get an index of 1 and so index theory by itself does not seem able to rule this out.



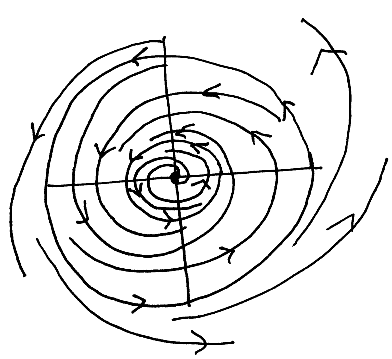


6.8.8 a) The sketch of the arrangement of the cycles is shown below.

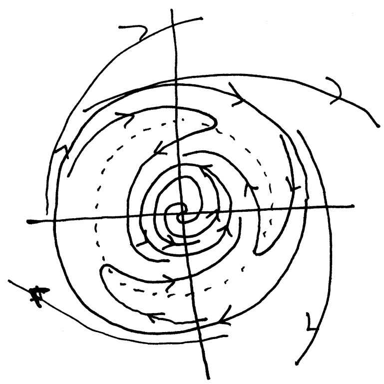


b) The sum of the index of the points in C_3 is the sum of the index of the points in C_1 (which must $= 1$ since it is a closed orbit) and the sum of the index of the points in C_2 (which also must $= 1$ since it is a closed orbit) annd the sum of the indices of the points inside of C_3 and outside C_1 and C_2. Since C_3 is a closed orbit the sum of the indices inside C_3 must $= 1$, but the sum of the indices inside C_1 and C_2 combined is $= 2$; so there must be at least one other fixed point to make the sum possible.

7.1.2 The angle is constantly increasing, moving counterclockwise. Looking at the function $\dot{r}$ we see that the function is positive for $0 < r < 1$, negative for $1 < r < 3$, and positive for $3 < r$. So we will spiral into the circle at $r = 1$ and spiral away from the circle at $r = 3$ and the origin, giving us the following picture:



7.1.3 The angle is increasing when $r < \sqrt{2}$ (i.e., moving counterclockwise) and is decreasing when $r > \sqrt{2}$ (i.e., moving clockwise). Looking at the function $\dot{r}$ we see that the function is positive for $0 < r < 1$, negative for $1 < r < 2$, and positive for $2 < r$. So we will spiral into the circle at $r = 1$ and spiral away from the circle at $r = 2$ and the origin, giving us the following picture:



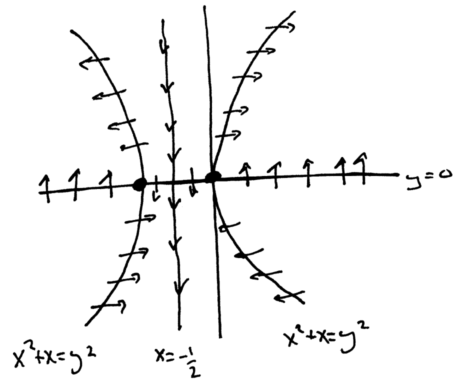
7.2.6 a) We have that $-\partial V/\partial x = y^2 + y \cos x$ and $-\partial V/\partial y = 2xy + \sin x$. Treating y as a constant and integrating the first expression we see that $V = -y^2 x - y \sin x + h(y)$ for some function $h(y)$. Taking the derivative of this with respect to y we see that $\partial V/\partial y = -2yx - \sin x + h'(y) = -2xy - \sin x$. It follows that $h'(y) = 0$ and so we can take $h(y) = 0$. Therefore we have that $V = -y^2 x - y \sin x$.

b) We have that $-\partial V/\partial x = 3x^2 - 1 - e^{2y}$ and $-\partial V/\partial y = -2xe^{2y}$. Treating y as a constant and integrating the first expression we see that $V = -x^3 + x + xe^{2y} + h(y)$ for some function $h(y)$. Taking the derivative of this with respect to y we see that $\partial V/\partial y = 2xe^{2y} + h'(y) = 2xe^{2y}$. It follows that $h'(y) = 0$ and so we can again take $h(y) = 0$. Therefore we have that $V = -x^3 + x + xe^{2y}$.

7.2.7 a) We have $\dot{x} = f(x, y) = y + 2xy$ and $\dot{y} = g(x, y) = x + x^2 - y^2$. Taking derivatives we have $\partial f/\partial y = 1 + 2x = \partial g/\partial x$. So this is a gradient system.

b) We have that $\partial V/\partial x = -y - 2xy$, now integrating we have that $V = -xy - x^2 y + h(y)$ for some function $h(y)$. Taking the derivative now with respect to y tells us that we need $\partial V/\partial y = -x - x^2 + h'(y) = -x - x^2 + y^2$. So we need $h'(y) = y^2$, so we can take $h(y) = \frac{1}{3}y^3$, giving $V = -xy - x^2 y + \frac{1}{3}y^3$.

c) The phase portrait is shown below.


y=0
x^2+x=y^2
x=-1/2
x^2+x=y^2