Homework 4 solutions

7.2.12 We have $\dot{x} = -x + 2y^3 - 2y^4$ and $\dot{y} = -x - y + xy$. Following the hint we are looking for a, m and n so that $V = x^m + ay^n$ is a Liapunov function. So we need such a function to satisfy $V > 0$ (which holds away from $(0,0)$ as long as m and n are even and a is positive), and also satisfies $\dot{V} < 0$. Computing we have that

$$\begin{aligned}\dot{V} &= mx^{m-1}\dot{x} + any^{n-1}\dot{y} = mx^{m-1}(-x + 2y^3 - 2y^4) + any^{n-1}(-x - y + xy) \\ &= \underbrace{-mx^m - any^n}_{\leq 0} + 2mx^{m-1}y^3 - 2mx^{m-1}y^4 - anxy^{n-1} + anxy^n\end{aligned}$$

Looking at the last four terms we want to be able to choose m and n so that we can (hopefully) cancel everything, i.e., we would choose $m = 2$ so that all the terms have x^1 and $n = 4$ so that all the terms have either y^3 or y^4. Making this substitution we now have

$$\dot{V} = -mx^m - any^n + (4 - 4a)xy^3 + (-4 + 4a)xy^4.$$

Clearly, we should choose $a = 1$, so that our Liapunov function is $V = x^2 + y^4$, and hence has no periodic solutions.

7.2.13 To apply Dulac's criterion we need to check that $\nabla\cdot(g\dot{\mathbf{x}})$ is either always positive or always negative. We have that $g = 1/(N_1N_2)$, $\dot{N}_1 = r_1N_1(1 - N_1/K_1) - b_1N_1N_2$ and $\dot{N}_2 = r_2N_2(1 - N_2/K_2) - b_2N_1N_2$. So we have

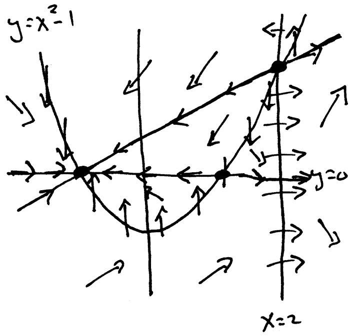
$$\begin{aligned}\nabla\cdot(g\dot{\mathbf{x}}) &= \frac{\partial}{\partial N_1}\Big(\frac{r_1(1 - N_1/K_1)}{N_2} - b_1\Big) + \frac{\partial}{\partial N_2}\Big(\frac{r_2(1 - N_2/K_2)}{N_1} - b_2\Big) \\ &= -\frac{r_1}{K_1N_2} - \frac{r_2}{K_2N_1} \\ &< 0.\end{aligned}$$

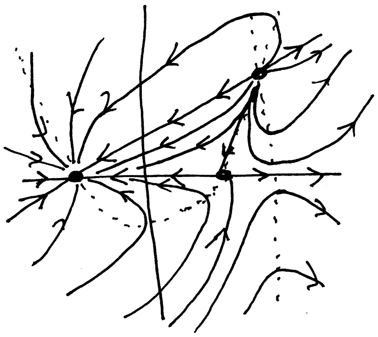
Since the region $N_1 > 0$, $N_2 > 0$ is connected and $\dot{\mathbf{x}}$ is smooth, then it now follows by Dulac's criterion that there are no closed orbits.

7.2.14 a) We have $\dot{x} = x^2 - y - 1$ and $\dot{y} = y(x-2)$. From the second equation we know that fixed points must occur when either $y = 0$ or $x = 2$, putting these into the first equation we get the points $(\pm 1, 0)$ and $(2, 3)$. To classify them we can consider the Jacobians $\left[\begin{smallmatrix} 2x & -1 \\ y & x-2 \end{smallmatrix}\right]$. At $(-1, 0)$ the Jacobian is $\left[\begin{smallmatrix} -2 & -1 \\ 0 & -3 \end{smallmatrix}\right]$ with eigenvalues -2 and -3 so that it is a stable point. At $(1, 0)$ the Jacobian is $\left[\begin{smallmatrix} 2 & -1 \\ 0 & -1 \end{smallmatrix}\right]$ with eigenvalues 2 and -1 so that this is a saddle point. Finally, at $(2, 3)$ the Jacobian is $\left[\begin{smallmatrix} 4 & -1 \\ 3 & 0 \end{smallmatrix}\right]$, which has characteristic equation $\lambda^2 - 4\lambda + 3$ and so the eigenvalues are 1 and 3 showing this is an unstable point.

b) On the line going through $(-1,0)$ and $(1,0)$ we have the x-axis and it is simple to see that the trajectories on the axis stay on the axis. So no closed orbit can hit the axis. Similarly consider the line between the point $(-1,0)$ and $(2,3)$. This line satisfies $y = x+1$, if we plug this into $\dot{x}$ and $\dot{y}$ we get that $\dot{x} = \dot{y} = x^2 - x - 2$. In particular the direction of the vector field on the line is in the direction of the line. In other words the line $y = x + 1$ is also a trajectory, so by the same logic no closed orbit can hit this line. But now we know that a closed orbit must contain some fixed point but any curve enclosing one of our fixed points would have to hit either the x-axis or $y = x + 1$ which is impossible. So there must be no closed orbit.

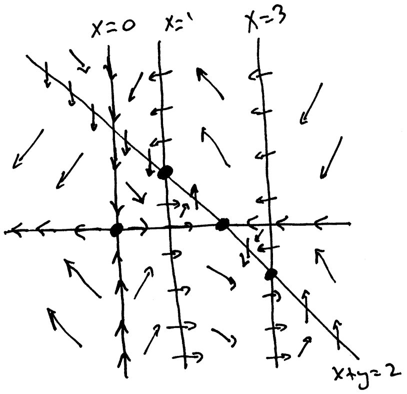
c) Plotting the null-clines and then using this to get the phase portrait we get the following picture.

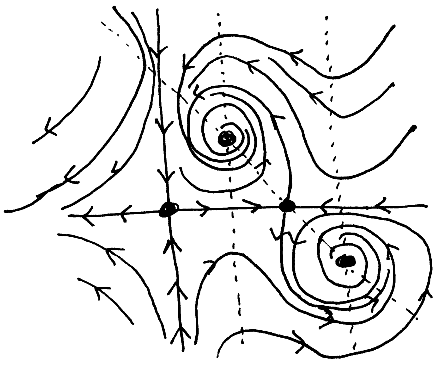




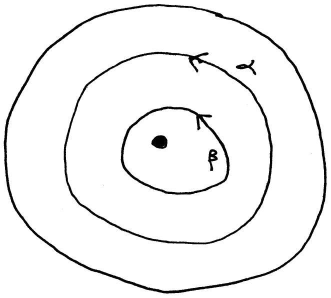
7.2.15 a) We have $\dot{x} = x(2-x-y)$ and $\dot{y} = y(4x - x^2 - 3)$. From the first equation we have that either $x = 0$ or $x + y = 2$ while from the second equation we have that either $y = 0$ or $x^2 - 4x + 3 = (x-3)(x-1) = 0$, i.e., $x = 1$ or 3. Combining these two set of solutions in all possible ways we get $(0,0)$, $(2,0)$, $(1,1)$ and $(3,-1)$. To classify them we look at the Jacobian $\left(\begin{smallmatrix} 2-2x-y & -x \\ 4y-2xy & 4x-x^2-3 \end{smallmatrix}\right)$. At $(0,0)$ the Jacobian is $\left(\begin{smallmatrix} 2 & 0 \\ 0 & -3 \end{smallmatrix}\right)$ with eigenvalues 2 and -3 so that $(0,0)$ is a saddle point. At $(2,0)$ the Jacobian is $\left(\begin{smallmatrix} -2 & -2 \\ 0 & 1 \end{smallmatrix}\right)$ with eigenvalues -2 and 1 so that $(2,0)$ is a saddle point. At $(1,1)$ the Jacobian is $\left(\begin{smallmatrix} -1 & -1 \\ 2 & 0 \end{smallmatrix}\right)$ with characteristic equation $\lambda^2 + \lambda + 2$ which has eigenvalues $(-1 \pm \sqrt{7}i)/2$ so $(1,1)$ is a stable spiral. Finally, at $(3,-1)$ the Jacobian is $\left(\begin{smallmatrix} -3 & -3 \\ 2 & 0 \end{smallmatrix}\right)$ with characteristic equation $\lambda^2 + 3\lambda + 6$ which has eigenvalues $(-3 \pm 3\sqrt{3}i)/2$ so that $(3,-1)$ is also a stable spiral.

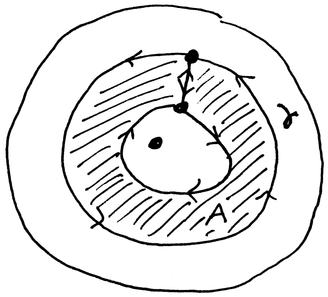
b) Plotting the null-clines and then using this to get the phase portrait we get the following picture.





7.2.17 To show there is at most one we only need show that there are not two. First off by Dulac's criterion any closed curve must contain the hole. So now suppose that there are two closed orbits, α and β, as shown in the picture below on the left which both enclose the hole.





Now the problem of applying Green's Theorem (which was used in the proof for Dulac's criterion) to any one of these curves is that there is the hole. So what we will do is create a new curve γ which is found by starting at a point on α, going all the way around, bridging to β, going all the way around in the opposite direction and then going back over the bridge to where we started. This will enclose an area A (as seen above on the picture on the right) which does not have holes and to which we can apply Green's Theorem. So we now have

$$\iint_A \nabla\cdot(g\dot{\mathbf{x}})dA = \oint_\gamma g\dot{\mathbf{x}}\cdot\mathbf{n}d\ell = \oint_\alpha g\dot{\mathbf{x}}\cdot\mathbf{n}d\ell - \oint_\beta g\dot{\mathbf{x}}\cdot\mathbf{n}d\ell.$$

Note, in going from the second to the third term we break the integral into parts and notice that since we cross the bridge once each way the two terms will cancel out. Now by assumption since $\nabla\cdot(g\dot{\mathbf{x}})$ is either always positive or negative this must be 0 the term on the left is nonzero. On the other hand since α and β are closed curves then $\dot{\mathbf{x}} \cdot \mathbf{n} = 0$ and so the integrals on the right are both 0. This is of course impossible. So our assumption that there were two closed curves is false and so there can be at most one.

* If we have $\dot{\mathbf{x}} = -\nabla V$ then the Jacobian would be

$$-\begin{pmatrix} \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}V\right) & \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}V\right) \\ \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}V\right) & \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}V\right) \end{pmatrix}.$$

Since $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}V\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}V\right)$ the matrix will be symmetric.

Given a symmetric matrix $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ the characteristic equation is $\lambda^2 - (a+c)\lambda + (ac - b^2)$. Putting this into the quadratic formula we have that the eigenvalues are

$$\frac{(a+c) \pm \sqrt{(a+c)^2 - 4(ac-b^2)}}{2} = \frac{(a+c) \pm \sqrt{a^2 + 2ac + c^2 - 4ac + 4b^2}}{2} = \frac{(a+c) \pm \sqrt{(a-c)^2 + 4b^2}}{2}.$$

In particular since the term inside the square root must be nonnegative the eigenvalues must be real. Showing there can be no spiral.

This makes sense since gradients should point in the direction of greatest change. In particular if we had a bowl then the direction of greatest change is straight down the side of the bowl (i.e., like a stable point), not slowly circling in along the outside (i.e., like a stable spiral).

** Plotting the null-clines and then using this to get the phase portrait we get the following picture.