Homework 5 solutions

7.3.1 a) The Jacobian at the origin is $\left(\begin{smallmatrix}1 & 1\\ -1 & 1\end{smallmatrix}\right)$, which has a characteristic equation $\lambda^2 - 2\lambda + 2$. So the eigenvalues are $1+i$ and $1-i$ which would make this point an unstable spiral.

b) We have

$$\begin{aligned} r\dot{r} = x\dot{x} + y\dot{y} &= x\big(x - y - x(x^2+5y^2)\big) + y\big(x + y - y(x^2+y^2)\big) \\ &= (x^2+y^2) - (x^2+y^2)^2 - 4x^2y^2 = r^2 - r^4 - 4r^4\cos^2(\theta)\sin^2(\theta) \\ &= r^2 - r^4\big(1+\sin^2(2\theta)\big), \end{aligned}$$

so that $\dot{r} = r - r^3\big(1+\sin^2(2\theta)\big)$. Similarly, we have

$$\begin{aligned} \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} &= \frac{x\big(x + y - y(x^2+y^2) - y\big(x - y - x(x^2+5y^2)\big)}{r^2} \\ &= \frac{(x^2+y^2) + 4xy^3}{r^2} = \frac{r^2 + 4r^4\cos(\theta)\sin^3(\theta)}{r^2} = 1 + 4r^2\cos(\theta)\sin^3(\theta). \end{aligned}$$

c) We have that $r - 2r^3 < \dot{r} < r - r^3$. In particular for $r < \sqrt{1/2}$ that $r - 2r^3 > 0$ and so the trajectories have radially outward components. Further for $r > \sqrt{1/2}$ it is easy to see that at $\theta = \pi/2$ that $\dot{r} > 0$ showing this is the maximumal such r_1.

d) Similarly as in the last part we see that if $r > 1$ that $r - r^3 < 0$ and so the trajectories have radially inward components. Further for $r < 1$ it is easy to see that at $\theta = 0$ that $\dot{r} < 0$ showing this is the minimal such r_2.

e) Any orbit starting between $\sqrt{1/2} < r < 1$ will stay in that range. By the Poincare-Bendixson Theorem it will follow that there must be a closed orbit if we can show that there are no fixed points (all other conditions are now easily satisfied). If there were a fixed point we would have to have that $1/r^2 = 1+\sin^2(\theta)$ and also have $1/r^2 = -4\cos(\theta)\sin^3(\theta)$. So a fixed point would occur when $1 + 4\cos(\theta)\sin^3(\theta) + \sin^2(\theta) = 0$, however it can be checked (i.e., graphically) that this does not happen. Therefore a closed orbit must exist.

7.3.3 If we try converting this to polar coordinates we have

$$\begin{aligned} \dot{r} = \frac{x\dot{x} + y\dot{y}}{r} &= \frac{x(x - y - x^3) + y(x + y - y^3)}{r} = \frac{x^2 + y^2 - x^4 - y^4}{r} \\ &= \frac{r^2 - r^4\cos^4(\theta) - r^4\sin^4\theta}{r} = r - r^3(\cos^4(\theta) + \sin^4(\theta)). \end{aligned}$$

Similarly we have

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = \frac{x(x+y-y^3) - y(x-y-x^3)}{r^2} = \frac{(x^2+y^2) + xy(x^2-y^2)}{r^2}$$
$$= \frac{r^2 + r^4\frac{1}{2}\sin(2\theta)\cos(2\theta)}{r^2} = 1 + \frac{1}{4}r^2\sin(4\theta).$$

Since $\frac{1}{2} \le \cos^4(\theta) + \sin^4(\theta) \le 1$ then similarly as to the previous problem when $r < 1$ we spiral out and when $r > \sqrt{2}$ we spiral in. So any orbit starting in $1 < r < \sqrt{2}$ stays in that region. Since there are no fixed points (i.e., such a fixed point would require $1/2 \ge \cos^4(\theta) + \sin^4(\theta) = -\sin(4\theta)/4 \le 1/4$, which of course is impossible) then by the Poincare-Bendixson Theorem it follows that there is a closed orbit.

7.3.4 a) The Jacobian at the origin is $\left(\begin{smallmatrix} 1 & 2 \\ -1/2 & 1 \end{smallmatrix}\right)$, which has a characteristic equation of $\lambda^2 - 2\lambda + 2$. This has eigenvalues of $1 \pm i$ and so the origin is an unstable spiral.

b) Clearly we have that on the ellipse $V = 0$ and away from these points $V > 0$. To show that all orbits approach the ellipse we only need to show that any point besides the origin and points on the ellipse have $\dot{V} < 0$. So we have

$$\begin{aligned}
\dot{V} &= 2(1-4x^2-y^2)(-8x\dot{x} - 2y\dot{y}) \\
&= 2(1-4x^2-y^2)\big(-8x(x(1-4x^2-y^2) - \frac{1}{2}y(1+x)) \\
&\qquad - 2y(y(1-4x^2-y^2) + 2x(1+x))\big) \\
&= 2(1-4x^2-y^2)\big((-8x^2-2y^2)(1-4x^2-y^2)\big) \\
&= -4(1-4x^2-y^2)^2(4x^2+y^2).
\end{aligned}$$

Clearly this is nonpositive and the only way it can be 0 is to be at the origin or on the ellipse.

7.3.7 a) In polar coordinates we would have

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \frac{x(y + ax(1-2b-r^2)) + y(-x + ay(1-r^2))}{r}$$
$$= \frac{a\big((x^2+y^2)(1-r^2) - 2bx^2\big)}{r} = \frac{a\big(r^2(1-r^2) - 2br^2\cos^2(\theta)\big)}{r}$$
$$= a\big(r(1-2b\cos^2(\theta)) - r^3\big),$$

and

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = \frac{x(-x + ay(1-r^2)) - y(y + ax(1-2b-r^2))}{r^2}$$
$$= \frac{-y^2 - x^2 + 2baxy}{r^2} = -\sin^2(\theta) - \cos^2(\theta) + 2ba\sin(\theta)\cos(\theta) = -1 + ab\sin(2\theta).$$

b) Repeating the same techniques as in previous problems we note that

$$(1-2b)r - r^3 < r(1-2b\cos^2(\theta)) - r^3 < r - r^3.$$

In particular, for $r < \sqrt{1-2b}$ we have that $\dot{r} > 0$ so that it is spiraling out, why for $r > 1$ we have that $\dot{r} < 0$ so that it is spiraling in. Since $\dot{\theta} \neq 0$ there are no fixed points in the interval between $\sqrt{1-2b}$ and 1, so by the Poincare-Bendixson Theorem there is at least one fixed orbit in that interval. Note that the period is found by

$$\int_0^{2\pi} \frac{dt}{\dot{\theta}} = \int_0^{2\pi} \frac{dt}{-1+ab\sin(2\theta)},$$

since this last integral is independent of r (i.e., which closed orbit we are on) then all closed orbits would have to have the same period.

c) In the case that $b = 0$ the equations reduce to $\dot{r} = a(r - r^3)$ and $\dot{\theta} = -1$. This has a closed orbit at $r = 1$ which orbits at a steady rate. For $0 < r < 1$ we have that $\dot{r} > 0$ and for $1 < r$ we have that $\dot{r} < 0$ so that away from the orbit we will move in towards the orbit, i.e., it is the uniques closed orbit.

7.3.9 a) We let $r(\theta) = 1 + \mu r_1(\theta) + O(\mu^2)$, and so $r'(\theta) = \mu r_1'(\theta) + O(\mu^2)$. Making these substitutions we have that $dr/d\theta = r(1-r^2) + \mu r\cos(\theta)$ becomes (with a little manipulation)

$$\begin{aligned} &\mu r_1'(\theta) + O(\mu^2) \\ &= (1+\mu r_1(\theta)+O(\mu^2))\big(1-(1+\mu r_1(\theta)+O(\mu^2))^2\big) + \mu\big(1+\mu r_1(\theta)+O(\mu^2)\big)\cos(\theta) \\ &= -2\mu r_1(\theta) + \mu\cos(\theta) + O(\mu^2) \end{aligned}$$

Ignoring the $O(\mu^2)$ terms and dividing by μ gives us $r_1'(\theta) = -2r_1(\theta) + \cos(\theta)$. To solve this we rearrange and get $r_1'(\theta)+2r_1(\theta) = \cos(\theta)$ which has an integrating factor of $e^{2\theta}$ and so we have

$$\big(r_1 e^{2\theta}\big)' = \cos(\theta)e^{2\theta},$$

and so integrating both sides we have

$$r_1 e^{2\theta} = \frac{2}{5}\cos(\theta)e^{2\theta} + \frac{1}{5}\sin(\theta)e^{2\theta} + C,$$

so that

$$r_1(\theta) = \frac{2}{5}\cos(\theta) + \frac{1}{5}\sin(\theta) + \underbrace{Ce^{-2\theta}}_{=0}.$$

(The reason that we need the last term to be 0 is so that the function will be periodic.) So we now have that

$$r(\theta) \approx 1 + \mu\Big(\frac{2}{5}\cos(\theta) + \frac{1}{5}\sin(\theta)\Big)$$

b) For $f(\theta) = 2\cos(\theta) + \sin(\theta)$ the maximum and minimum occur when $f'(\theta) = 0$ or when $-2\sin(\theta) + \cos(\theta) = 0$, or when $\tan(\theta) = 1/2$. It follows from this that the maximum value for $f(\theta)$ is $\sqrt{5}$ (where $\cos(\theta) = 2/\sqrt{5}$ and $\sin(\theta) = 1/\sqrt{5}$) while the minimum value is $-\sqrt{5}$ (where $\cos(\theta) = -2/\sqrt{5}$ and $\sin(\theta) = -1/\sqrt{5}$).

   So in particular we have that the maximum and minimum values of our approximate orbit are $1 + \sqrt{5}\mu/5 \approx 1 + (0.444721\ldots)\mu$ and $1 - \sqrt{5}\mu/5 \approx 1 - (0.444721\ldots)\mu$

   On the other hand we have that for $\mu \ll 1$ we have that $\sqrt{1+\mu} \approx 1 + (0.5)\mu$ and $\sqrt{1-\mu} \approx 1 - (0.5)\mu$ so that our approximate orbit will lie inside the annulus $\sqrt{1-\mu} < r < \sqrt{1+\mu}$ as expected.

* Since any small open interval must contain both rational and irrational numbers neither the rationals or irrationals are open (and hence also closed) in $\mathbb{R}$.

** (a) This has two accumulation points, 1 and -1. (To see this consider what happens for the cases when n is even and when n is odd.)

   (b) This sequence repeats the same six points over and over again $(1, 0)$, $(1/2, \sqrt{3}/2)$, $(-1/2, \sqrt{3}/2)$, $(-1, 0)$, $(-1/2, -\sqrt{3}/2)$ and $(1/2, -\sqrt{3}/2)$, and so these six points are the accumulation points.

   (c) The accumulation points are the entire unit circle, i.e., for arbitrarily large n we can get arbitrarily close to any point on the circle.