Homework 6 solutions

(1) Starting with

$$\dot{x} = \frac{1}{8} - x + x^2 y \qquad \text{and} \qquad \dot{y} = \frac{1}{2} - x^2 y$$

we see that at the fixed point we must have $x^2 y = 1/2$ implying that $x = 1/2 + 1/8 = 5/8$, this then implies that $y = 1/(2x^2) = 1/(50/64) = 32/25$. Now looking at the stability of this fixed point we would consider the Jacobian which is

$$\left(\begin{array}{cc} -1+2xy & x^2 \\ -2xy & -x^2 \end{array} \right)\Bigg|_{(x,y)=(\frac{5}{8},\frac{32}{25})} = \left(\begin{array}{cc} \frac{3}{5} & \frac{25}{64} \\ -\frac{8}{5} & -\frac{25}{64} \end{array} \right).$$

Calculating, we have that the eigenvalues of this matrix are $(67 \pm 3i\sqrt{17279})/640$, in particular they are imaginary with positive real part so that the point is an unstable spiral. The key to this is that in any neighborhood we move out of that point.

It will then follows that if we show that the region is positively invariant then take the region minus a small ball around our fixed point and apply the Poincaré-Bendixson Theorem to get that any trajectory must approach a limit cycle, and hence there must be a closed orbit.

So now we turn to the edges of our domain $\{(x,y) : x \geq 1/16, 0 \leq y \leq 128, x+y \leq 130\}$. First if we area at $x = 1/16$ then $\dot{x} = 1/16 + x^2 y > 0$ along that edge of the domain and so we will flow into the domain. Similarly if $y = 0$ then $\dot{y} = 1/2 > 0$ so that we will again flow into the domain. Now suppose that $y = 128$ and $1/16 \leq x \leq 2$ (i.e., at $y = 128$ there is only a small strip to consider), in this case we have that $\dot{y} = (1/2) - 128x^2 \leq 0$ and so we still flow in. Finally, we turn to the line $x + y = 130$, with $2 \leq x \leq 130$. We note in particular that on this line

$$\begin{aligned} \dot{x} &= (1/8) - x + x^2(x - 130) < 0 \\ \dot{y} &= (1/2) - x^2(x - 130). \end{aligned}$$

In particular we have that the slope of intersection along this line will satisfy,

$$-1 < \frac{\dot{y}}{\dot{x}} = \frac{(1/2) - x^2(x-130)}{(1/8) - x + x^2(x-130)} < 0$$

combined with knowing that $\dot{x} < 0$ we must cross into the region if we are on the line.

(2) (a) Starting with

$$\dot{x} = x\left(1 - \frac{x}{30} - \frac{y}{x+10}\right), \qquad \text{and} \qquad \dot{y} = y\left(\frac{x}{x+10} - \frac{1}{3}\right),$$

we have that at a fixed point either $y = 0$ or $x/(x+10) = 1/3$. In the first case we then have that either $x = 0$ or $x = 30$ to ensure that $\dot{x} = 0$. In the second case we can simplify and find that $3x = x + 10$ or $2x = 10$ or $x = 5$, since $x = 5$ we must have that $1 - (1/6) - (y/15) = 0$ or, multiplying by 30, $25 - 2y = 0$ so that $y = 25/2$. Therefore out three fixed points are $(0, 0)$, $(30, 0)$ and $(5, 25/2)$. To classify them we look at the Jacobian, i.e.,

$$\begin{pmatrix} 1 - \frac{x}{15} - \frac{10y}{(x+10)^2} & \frac{-x}{x+10} \\ \frac{10y}{(x+10)^2} & \frac{x}{x+10} - \frac{1}{3} \end{pmatrix}..$$

At $(0, 0)$ the Jacobian is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1/3 \end{smallmatrix}\right)$ which has one positive and one negative eigenvalue so that $(0, 0)$ is a saddle. At $(30, 0)$ the Jacobian is $\left(\begin{smallmatrix} -1 & -3/4 \\ 0 & 5/12 \end{smallmatrix}\right)$, again with one positive and one negative eigenvalue we have that $(30, 0)$ is a saddle. Finally for $(5, 25/2)$ we have that the Jacobian is $\left(\begin{smallmatrix} 1/9 & -1/3 \\ 5/9 & 0 \end{smallmatrix}\right)$. In this last case the eigenvalues are $(1 \pm i\sqrt{119})/18$ which are complex with positive real part and so is an unstable spiral.

(b) When we are on the x-axis (i.e., $y = 0$) then we have $\dot{y} = 0$ and so we stay on the x-axis. Further for $0 \le x \le 50$ we have that the trajectories will move to $x = 30$. When we are on the y-axis (i.e., $x = 0$) then we have that $\dot{x} = 0$ and so we stay on the y axis. Further for $0 \le y \le 50$ we have that the trajectories move to $y = 0$. In particular we know that if we start on the axes in our domain then we stay on the axes in our domain.

It remains to check that we cannot cross out of our domain over the line $x + y = 50$. Note in particular that on this line

$$\begin{aligned} \dot{x} &= x\big(1 - \frac{x}{20} - \frac{50 - x}{x + 10}\big) = \frac{-x\big((x-25)^2 + 575\big)}{30(x+10)} < 0 \\ \dot{y} &= \frac{(50 - x)(2x - 10)}{3(x + 10)}, \end{aligned}$$

In particular we have that the slope of intersection along the line $x + y = 50$ is

$$\frac{\dot{y}}{\dot{x}} = \frac{10(50 - x)(2x - 10)}{-x\big((x-25)^2 + 575\big)}.$$

Minimizing this we discover that the minimal slope of the tangent line is $-3/4$ and occurs at $x = 20$, combined with the fact that $\dot{x} < 0$ this will imply that along the line $x + y = 50$ we move *into* the region, i.e., the region is invariant.

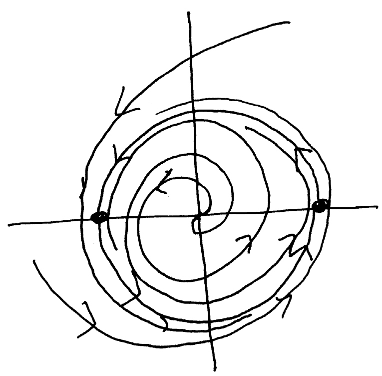
(c) Since the region is invariant, and we can put a small neighborhood around the fixed point $(5, 25/2)$ so that point move out of that region we can then apply Poincaré-Bendixson Theorem to show that inside our region, but away from our fixed point that the trajectory will approach a limiting cycle.

(3) (a) To be a limit point of a trajectory means that in every ϵ-neighborhood of that point there is some point in the trajectory. So in particular a limit point is any point so that we can find some sequence $t_1, t_2, \ldots, t_n, \ldots$ so that the $x(t_n)$ converges to that point. For comparison, an ω-limit point is a point so that there exists a sequence $t_1, t_2, \ldots, t_n, \ldots$ *where* $t_n \to \infty$ and where the $x(t_n)$ converge to that point.

So the definition of an ω-limit point is a more restrictive form of the definition of a limit point, so every ω-limit point is also a limit point.

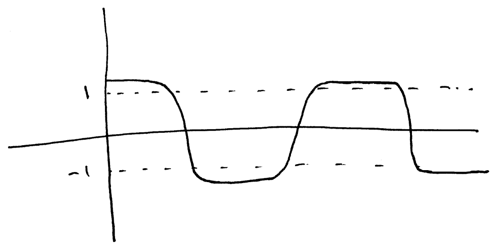
(b) This can easily happen. For example, take any point on any (continuous) trajectory then that point is a limit point of the trajectory but not necessarily an ω-limit point.

7.3.11 a) Starting with $\dot{r} = r(1 - r^2)[r^2 \sin^2(\theta) + (r^2 \cos^2(\theta) - 1)^2]$ and $\dot{\theta} = r^2 \sin^2(\theta) + (r^2 \cos^2(\theta) - 1)^2$ we first note that the fixed points occur when $r^2 \sin^2(\theta) + (r^2 \cos^2(\theta) - 1)^2 = 0$. Translating this into polar coordinates this becomes $y^2 + (x^2 - 1)^2 = 0$ which can only be 0 when $y = 0$ and $x^2 = 1$, i.e., $x = \pm 1$. So there are two fixed points, namely at $(x, y) = (\pm 1, 0)$. Away from the fixed points we have that $\dot{\theta} > 0$ so that we always rotate counterclockwise. In addition the other zeroes of $\dot{r}$ are 1 and $r = 1$. This gives us the following picture.



You might notice that this picture is very similar to $\dot{r} = r(1 - r^2)$ and $\dot{\theta} = 1$. This is no coincidence, in fact it *is* the same picture except the introduction of the two new fixed points. (One way to see this is to note that away from the fixed points that in both cases we have $dr/d\theta = r(1 - r^2)$). However the behavior of these two systems is different as should be noticed in part (b).

b) On the spiral when we are "close" to a fixed point (or x is close to ± 1) the quantity $r^2 \sin^2(\theta) + (r^2 \cos^2(\theta) - 1)^2$ is small. As a result we will move slowly, while if we are far from a fixed point we will move quickly. As a consequence instead of looking like a typical sinusoidal curve it will have a more of a box-like appearance as shown below (though the picture does not do it the justice it deserves).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