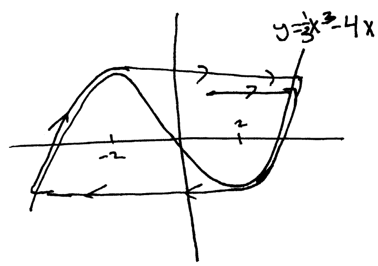
Homework 7 solutions

7.5.3 We have $\ddot{x} + k(x^2 - 4)\dot{x} + x = 1$ with $k \gg 1$. This is very similar to the van der Pol equation and so we will mimic the analysis done for that. First we parametrize by letting $F(x) = \frac{1}{3}x^3 - 4x$ so that if we let $w = \dot{x} + kF(x)$, then $\dot{w} = \ddot{x} + k(x^2 - 4) = 1 - x$. If we now substitute $w = ky$ then this can be rewritten as $\dot{x} = k\big(y - F(x)\big)$ and $\dot{y} = \frac{1}{k}(1 - x)$. Now if we start away from $(1, -11/3)$ (i.e., the fixed point) then we claim that the trajectory in the Liénard plane has the following shape.



Note in particular that the extreme values occur at ± 2 with values $\mp\frac{16}{3}$. These same values occur at ± 4 (to get this set $\frac{1}{3}x^3 - 4x = \frac{16}{3}$ which can then be rewritten as $x^3 - 12x - 16 = 0$, since we already know -2 is a root we can do long division to reduce it to a quadratic and then solve).

We know that the time is well approximated by the time spent "crawling" along the cubic. So we have that

$$T \approx \int_{t_A}^{t_B} dt + \int_{t_C}^{t_D} dt,$$

where between t_A and t_B we are on the left hand side crawling while between t_C and t_D we are on the right hand side crawling. Along this crawl we are near the cubic so that $\frac{dy}{dt} \approx F'(x)\frac{dx}{dt} = (x^2 - 4)\frac{dx}{dt}$, but since $\frac{dy}{dt} = (1 - x)/k$ then we have that

$$\frac{dx}{dt} \approx \frac{1 - x}{k(x^2 - 4)} \qquad \text{or} \qquad dt \approx \frac{k(x^2 - 4)}{1 - x}\, dx$$

From the above discussions we have that $t_A = -4$, $t_B = -2$, $t_C = 4$ and $t_D = 2$. So combining everything we have

$$\begin{aligned}
T &\approx \int_{-4}^{-2} \frac{k(x^2-4)}{1-x}\,dx + \int_{4}^{2} \frac{k(x^2-4)}{1-x}\,dx \\
&= -k\int_{-4}^{-2} \big(x+1+\frac{3}{1-x}\big)\,dx - k\int_{4}^{2} \big(x+1+\frac{3}{1-x}\big)\,dx \\
&= -k\Big(\big(\frac{1}{2}x^2+x-3\ln|1-x|\big)\Big|_{-4}^{-2} + \big(\frac{1}{2}x^2+x-3\ln|1-x|\big)\Big|_{4}^{2}\Big) \\
&= -k\big((2-2-3\ln(3)) - (8-4-3\ln(5)) + (2+2-3\ln(1)) - (8+4-3\ln(3))\big) \\
&= \big(12-3\ln(5)\big)k.
\end{aligned}$$

7.5.4 a) If we let $f(x) = -1$ for $|x| < 1$ and $f(x) = 1$ for $|x| \geq 1$ then if $F'(x) = f(x)$ with $F(0) = 0$ it must be that it has the form

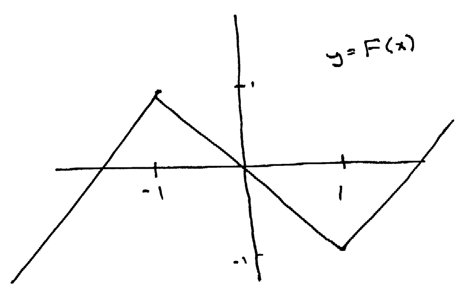
$$F(x) = \begin{cases} x+A & \text{if } x \leq -1, \\ -x+B & \text{if } |x| \leq 1, \\ x+C & \text{if } x \geq 1, \end{cases}$$

where A, B and C are constants to be determined. Since $F(0) = 0$ that implies that $B = 0$ and then to be continuous at 1 we need $-1 = 1 + C$ so that $C = -2$ and similarly to be continuous at -1 we need $1 = -1 + B$ so that $B = 2$. So we have the following form for $F(x)$.

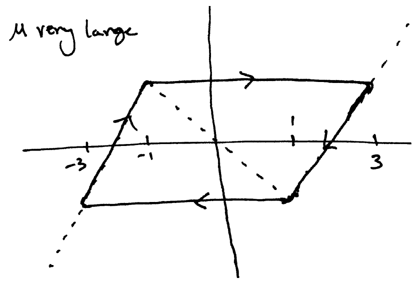
$$F(x) = \begin{cases} x+2 & \text{if } x \leq -1, \\ -x & \text{if } |x| \leq 1, \\ x-2 & \text{if } x \geq 1. \end{cases}$$

So now let $w = \dot{x} + \mu F(x)$ so that $\dot{w} = \ddot{x} + \mu f(x)\dot{x} = -x$. If we now let $w = \mu y$ then this can be rearranged as $\dot{x} = \mu\big(y - F(x)\big)$ and $\dot{y} = -x/\mu$.

b) The nullclines are shown in the picture below. (Sketched in the Liénard plane.)



c) For $\mu \gg 1$ then we will move quickly horizontally while we are away from $F(x)$ (i.e., since it has the μ term attached to $\dot{x}$). On the other hand after we cross we will move slowly down the side of the curve until we turn the corner and can start our horizontal movement (i.e., since it has the $1/\mu$ term attached to $\dot{y}$). This indicates that the system will have a relaxation oscillation. The sketch of the limit cycle is shown below.



d) To estimate the time we need to estimate the time spent on the slow crawls along the sides. These happen for $1 \leq x \leq 3$ and $-1 \leq x \leq -3$. In both cases we have that $\frac{dy}{dt} \approx \frac{dx}{dt}$ but we also have that $\frac{dy}{dt} = -x/\mu$. Combining we have that $\frac{dx}{dt} = -x/\mu$ or $dt = -\frac{\mu}{x}dx$. Now setting up our integrals we have that

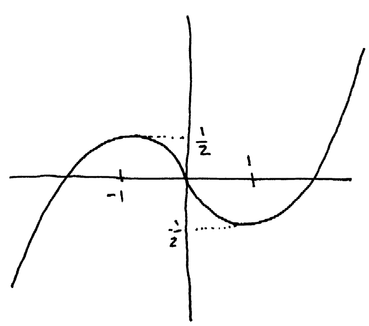
$$T \approx \int_3^1 -\frac{\mu}{x}\,dx + \int_{-3}^{-1} -\frac{\mu}{x}\,dx = 2\mu \int_1^3 \frac{dx}{x} = 2\mu \ln x \Big|_1^3 = (2\ln 3)\mu.$$

7.5.5 First for $f(x) = |x| - 1$ we can break this up and find a peicewise antiderivative $F(x)$ which satisfies $F(0) = 0$, namely the following function:

$$F(x) = \begin{cases} -\frac{1}{2}x^2 - x & \text{if } x \leq 0, \\ \frac{1}{2}x^2 - x & \text{if } x \geq 0. \end{cases}$$

Now we proceed as before by letting $w = \dot{x} + \mu F(x)$ so that $\dot{w} = \ddot{x} + \mu(|x| - 1)\dot{x} = -x$. If we now let $w = \mu y$ then this can be rewritten as the system $\dot{x} = \mu\big(y - F(x)\big)$ and $\dot{y} = -x/\mu$.

If we now plot the nullclines we get the following graph.



In particular the extremum values are $\pm\frac{1}{2}$. We have that $\frac{1}{2}x^2 - x = \frac{1}{2}$ is equivalent to $x^2 - 2x - 1 = 0$ which can be solved using the quadratic equation to give $x = 1 + \sqrt{2}$. Similarly on the other side we get $x = -1 - \sqrt{2}$. Now to find the approximate period we need to find the amount of time spent "crawling" along the sides. To do this we first note that $\frac{dy}{dt} \approx \big(|x| - 1\big)\frac{dx}{dt}$ and also we have that $\frac{dy}{dt} = -x/\mu$. Combining we have that $\big(|x| - 1\big)\frac{dx}{dt} \approx -x/\mu$, or

$$dt = \frac{\mu\big(1 - |x|\big)}{x}\,dx = \begin{cases} \dfrac{\mu(1+x)}{x} & \text{if } x \le 0, \\ \dfrac{\mu(1-x)}{x} & \text{if } x \ge 0. \end{cases}$$

Therefore we have

$$\begin{aligned}
T &\approx \int_{-1-\sqrt{2}}^{-1} \frac{\mu(1+x)}{x}\,dx + \int_{1+\sqrt{2}}^{1} \frac{\mu(1-x)}{x}\,dx \\
&= \mu\bigg(\big(x + \ln|x|\big)\Big|_{-1-\sqrt{2}}^{-1} + \big(-x + \ln|x|\big)\Big|_{1+\sqrt{2}}^{1}\bigg) \\
&= \mu\bigg(\big((-1) - (-1 - \sqrt{2} + \ln(1+\sqrt{2})\big) + \big((-1) - (-1 - \sqrt{2} + \ln(1+\sqrt{2})\big)\bigg) \\
&= 2\mu\big(\sqrt{2} - \ln(1+\sqrt{2})\big).
\end{aligned}$$

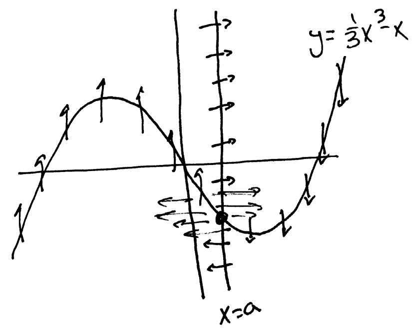
7.5.6 a) We again start by the same process as before. Namely let $F(x) = \frac{1}{3}x^3 - x$ and set $w = \dot{x} + \mu F(x)$ so that $\dot{w} = \ddot{x} + \mu(x^2 - 1)\dot{x} = a - x$. If we now let $w = \mu y$ then this can be rewritten as $\dot{x} = \mu\big(y - F(x)\big)$ and $\dot{y} = (a - x)/\mu$.

The fixed points would correspond to when $a - x = 0$ and $y - F(x) = 0$, so the unique fixed point occurs at $(a, F(a))$. Taking the Jacobian of the system we have that

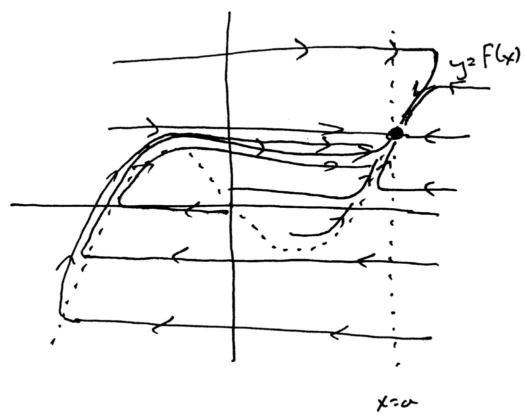
$$J_{(a,F(a))} = \begin{pmatrix} -\mu(a^2 - 1) & \mu \\ -1/\mu & 0 \end{pmatrix}.$$

The Jacobian has determinant 1, and so there are several possibilities, depending on the eigenvalues which in turn depend on the trace ($= -\mu(a^2 - 1)$). In the extreme cases we will have that $\mu^2(a^2 - 1)^2 \gg 0$ (except for $a = 1$, at which point we will have a center) and $-\mu(a^2 - 1) > 0$ for $|a| < 1$ which would give us unstable nodes and $-\mu(a^2 - 1) < 0$ for $|a| > 1$ which would give us stable nodes.

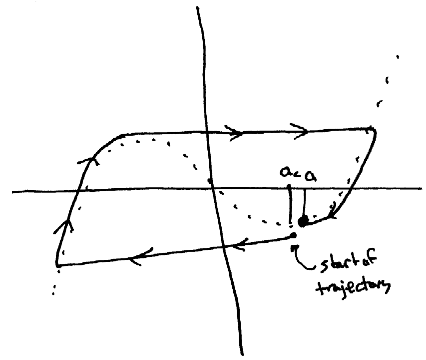
b) The plot of the nullclines are shown below. In particular, it is easy to see that if we start near $(a, F(a))$ and above $y = \frac{1}{3}x^3 - x$ we quickly move to the right away from the fixed point, while if we start below $y = \frac{1}{3}x^3 - x$ we quickly move to the left again away from the fixed point. In either case we move away and the point is unstable.



c) In part (b) we saw that when the fixed point was located where $|a| < 1$ it was unstable and it is easy to see that we have the same situation as in the (unbiased) van der Pol equation occurs, namely we have a limit cycle. We now claim that if $|a| > 1$ then there is a globally attracting fixed point at $(a, F(a))$. To see this consider a sketch of the phase portrait as shown below, note in particular that no matter where we start we end up at the fixed point.



d) If we have a fixed point just above a_c and then start our orbit a little below a_c then the trajectory has to make a long loop before returning to the fixed point. This situation is shown in the picture below.



7.6.2 a) We have that $\ddot{x} + (1+\epsilon)x = 0$. This is a well known system that has as its solution $x(t) = A\sin(\sqrt{1+\epsilon}\,t) + B\cos(\sqrt{1+\epsilon}\,t)$. Since our initial conditions are $x(0) = 1$ and $\dot{x}(0) = 0$ we have from the first that $B = 1$ and from the second that $\sqrt{1+\epsilon}A = 0$. Therefore the solution to the differential equation is $x(t) = \cos(\sqrt{1+\epsilon}\,t)$.

b) Start by letting $x = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + O(\epsilon^3)$. Then we have

$$\begin{aligned} 0 &= \ddot{x} + (1+\epsilon)x \\ &= \big(x_0''(t) + \epsilon x_1''(t) + \epsilon^2 x_2''(t) + O(\epsilon^3)\big) \\ &\qquad\qquad + (1+\epsilon)\big(x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + O(\epsilon^3)\big) \\ &= \big(x_0''(t) + x_0(t)\big) + \epsilon\big(x_1''(t) + x_1(t) + x_0(t)\big) + \epsilon^2\big(x_2''(t) + x_2(t) + x_1(t)\big) + O(\epsilon^3) \end{aligned}$$

The initial conditions translate into $x_0(0) = 1$ and $x_0'(0) = x_1(0) = x_1'(0) = x_2(0) = x_2'(0) = 0$.

Working our way up the solution to $x_0''(t) + x_0(t) = 0$ with $x_0(0) = 1$ and $x_0'(0) = 0$ is $x_0(t) = \cos(t)$. For the next term we have that $x_1''(t) + x_1(t) + \cos(t) = 0$ with $x_1(0) = x_1'(0) = 0$. Using the method of undetermined coefficients we have that a solution will have the form

$$x_1(t) = A\cos(t) + B\sin(t) + Ct\sin(t) + Dt\cos(t).$$

Since

$$x_1''(t) = -A\cos(t) - B\sin(t) + 2C\cos(t) - Ct\sin(t) - 2D\sin(t) - Dt\sin(t),$$

it follows that we need $2C + 1 = 0$ and $-2D = 0$, so that the solution will have the form $A\cos(t) + B\sin(t) - \frac{1}{2}t\sin(t)$. Now applying the initial conditions we see that $A = 0$ and $B = 0$, so that $x_1(t) = -\frac{1}{2}t\sin(t)$.

Going to the next step we have that $x_2''(t) + x_2(t) - \frac{1}{2}t\sin(t) = 0$ with $x_2(0) = x_2'(0) = 0$. For this we would guess a particular solution of the form as follows

$$x_2(t) = A\cos(t) + B\sin(t) + Ct\cos(t) + Dt\sin(t) + Et^2\cos(t) + Ft^2\sin(t),$$

Since

$$x_2''(t) = \cos(t)\big(-A - Ct + 2D + 2E - Et^2 + 4Ft\big) + \sin(t)\big(-B - 2C - Dt - 4Et + 2F - Ft^2\big)$$

we have that

$$x_2'' + x_2 - \frac{1}{2}t\sin(t) = \cos(t)\big(2D + 2E + 4Ft\big) + \sin(t)\big(-2C - 4Et + 2F - \frac{1}{2}t\big).$$

From this it follows that we need $F = C = 0$ and $E = -D = -\frac{1}{8}$ so that our solution has the form

$$x_2(t) = A\cos(t) + B\sin(t) + \frac{1}{8}t\sin(t) - \frac{1}{8}t^2\cos(t).$$

Our initial conditions that then give us that $A = B = 0$ so that

$$x_2(t) = \frac{1}{8}t\sin(t) - \frac{1}{8}t^2\cos(t).$$

Putting it all together we have

$$x(t) = \cos(t) - \epsilon\big(\frac{1}{2}t\sin(t)\big) + \epsilon^2\big(\frac{1}{8}t\sin(t) - \frac{1}{8}t^2\cos(t)\big) + O(\epsilon^3).$$

c) The terms $x_1(t)$ and $x_2(t)$ are secular terms (i.e., they blow up as $t \to \infty$). We might have expected to see this since the explicit solution and $x_0(t)$ have slightly different cycles and so over time they will start to diverge and so the other terms should be secular to compensate.

7.6.13 a) Starting with $\ddot{x} + x + \epsilon x^3 = 0$ we multiply both sides by $\dot{x}$ and then integrate to get $\ddot{x}\dot{x} + (x + \epsilon x^3)\dot{x} = 0$. This last form is convenient for integrating (i.e., similarly to when we did conservation of energy). In particular we have that $\frac{1}{2}(\dot{x})^2 + \frac{1}{2}x^2 + \frac{1}{4}\epsilon x^4 = C$. Since we know that $x(0) = a$ and $\dot{x} = 0$ we can solve for the constant explicitly and get that $C = \frac{1}{2}a^2 + \frac{1}{4}\epsilon a^4$. Multiplying by 2 and rearranging we have

$$(\dot{x})^2 = (a^2 - x^2) + \frac{\epsilon}{2}(a^4 - x^4) = (a^2 - x^2) + \frac{\epsilon}{2}(a^2 - x^2)(a^2 + x^2).$$

In particular we can take the square root and solve for dt and get

$$dt = \pm\frac{dx}{\sqrt{a^2 - x^2}\sqrt{1 + \frac{\epsilon}{2}(a^2 + x^2)}}.$$

The benefit of this is that we are trying to approximate a time which can be found by $\int dt$, we can now transform this into $\int (\text{stuff})\, dx$ where we know what the "stuff" is. The only thing left to do is to find the limits of integration. For $\epsilon \ll 1$ we would expect the solution to be close to that of $\ddot{x}+x=0$ which produces a circle in the phase plane. So we will use that circle as an approximation for our integral. Since our circle will have radius a we can integrate from 0 to a on the top and then multiply the result by 4. This gives us the following

$$T \approx 4\int_0^a \frac{dx}{\sqrt{a^2-x^2}\sqrt{1+\frac{\epsilon}{2}(a^2+x^2)}}.$$

b) To find the expansion in terms of ϵ we can start with a Taylor series which says that for z small,

$$\frac{1}{\sqrt{1+z}} = (1+z)^{-1/2} = 1 - \frac{1}{2}z + \frac{3}{8}z^2 + O(z^3).$$

Putting this into our integral (with $z = \frac{\epsilon}{2}(a^2+x^2)$) we have that

$$T \approx 4\int_0^a \frac{dx}{\sqrt{a^2-x^2}} - \epsilon\int_0^a \frac{(a^2+x^2)\,dx}{\sqrt{a^2-x^2}} + \frac{3}{8}\epsilon^2\int_0^a \frac{(a^2+x^2)^2\,dx}{\sqrt{a^2-x^2}} + O(\epsilon^3).$$

The first integral is simply $\arcsin(x/a)$. Evaluate it from 0 to a and multiply by 4 we get that the first term is 2π. This is what we would have expected since this should approximate the solution to $\ddot{x}+x=0$ which also has a period of 2π. For the other integrals we can use trig substitution (i.e., $x = a\sin(\theta)$) and do a lot of work, look up integral tables, or use a computer algebra system. Trusting in the latter we have that

$$\int_0^a \frac{(a^2+x^2)\,dx}{\sqrt{a^2-x^2}} = \frac{3}{4}a^2\pi \qquad \text{and} \qquad \int_0^a \frac{(a^2+x^2)^2\,dx}{\sqrt{a^2-x^2}} = \frac{19}{16}a^4\pi.$$

So we have that

$$T \approx 2\pi - \frac{3}{4}a^2\pi\epsilon + \frac{57}{128}a^4\pi\epsilon^2 + O(\epsilon^3).$$