Homework 8 solutions

7.6.19 a) By setting $\tau = \omega t$ we have that

$$\ddot{x} = \frac{d^2(x)}{d(t^2)} = \frac{d^2(x)}{d(\tau/\omega)^2} = \omega^2 \frac{d^2(x)}{d(\tau)^2} = \omega^2 x''.$$

So the Duffing equation $\ddot{x} + x + \varepsilon x^3 = 0$ becomes $\omega^2 x'' + x + \varepsilon x^3 = 0$.

b) We let $x(\tau, \varepsilon) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + O(\varepsilon^3)$ and $\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + O(\varepsilon^3)$. The first term in our differential equation from the previous part now becomes

$$\begin{aligned}\omega^2 x'' &= \left(1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + O(\varepsilon^3)\right)^2 \left(x_0''(\tau) + \varepsilon x_1''(\tau) + \varepsilon^2 x_2''(\tau) + O(\varepsilon^3)\right) \\ &= x_0''(\tau) + \varepsilon\big(x_1''(\tau) + 2\omega_1 x_0''(\tau)\big) + \varepsilon^2\left(\omega_1^2 x_0'' + 2\omega_1 x_1''(\tau) + x_2''(\tau)\right) + O(\varepsilon^3).\end{aligned}$$

In particular we have

$$\begin{array}{rcllr} \omega^2 x'' &=& x_0''(\tau) & + \varepsilon\big(x_1''(\tau) + 2\omega_1 x_0''(\tau)\big) & + O(\varepsilon^2) \\ x &=& x_0(\tau) & + \varepsilon x_1(\tau) & + O(\varepsilon^2) \\ \varepsilon x^3 &=& & + \varepsilon(x_0(\tau))^3 & + O(\varepsilon^2) \end{array}$$

In particular summing the columns we see that the constant term has $x_0''(\tau) + x_0(\tau) = 0$ while the term with coefficient ε is $x_1''(\tau) + 2\omega_1 x_0''(\tau) + x_1(\tau) + (x_0(\tau))^3 = 0$, which can be rewritten as $x_1''(\tau) + x_1(\tau) = -2\omega_1 x_0''(\tau) - (x_0(\tau))^3$.

c) Plugging in our initial conditions we have that

$$a = x(0) = x_0(0) + \varepsilon x_1(0) + \varepsilon x_2(0) + O(\varepsilon^3),$$
$$\text{and} \qquad 0 = x'(0) = x_0'(0) + \varepsilon x_1'(0) + \varepsilon x_2'(0) + O(\varepsilon^3).$$

Since both sides can be thought of as a power series in ε, then by matching coefficients we have that $a = x_0(0)$ while $0 = x_k(0)$ for $k \geq 1$ while $x_k'(0) = 0$ for all $k \geq 0$.

d) This is a well known differential equation with solution $x(\tau) = A\cos(\tau) + B\sin(\tau)$. To satisfy the initial conditions we need $x(0) = A = a$ and $x'(0) = B = 0$. So the solution for x_0 is $x_0(\tau) = a\cos(\tau)$.

e) Putting in out solution to x_0 we have

$$-2\omega_1 x_0''(\tau) - (x_0(\tau))^3 = 2a\omega_1 \cos(\tau) - a^3\cos^3(\tau).$$

Since

$$\begin{aligned}\cos^3(x) &= \left(\frac{e^{ix} + e^{-ix}}{2}\right)^3 = \frac{e^{i3x} + 3e^{ix} + 3e^{ix} + e^{-i3x}}{8} \\ &= \frac{1}{4}\frac{e^{i3x} + e^{-i3x}}{2} + \frac{3}{4}\frac{e^{ix} + e^{-ix}}{2} = \frac{1}{4}\cos(3x) + \frac{3}{4}\cos(x),\end{aligned}$$

then we have that

$$2a\omega_1\cos(\tau) - a^3\big(\frac{1}{4}\cos(3\tau) + \frac{3}{4}\cos(\tau)\big) = \big(2a\omega_1 - \frac{3}{4}a^3\big)\cos(\tau) - \frac{1}{4}a^3\cos(3\tau).$$

A secular term *must* occur is there is a $\cos(\tau)$ term on the left hand side, so we need to choose ω_1 so that this situation does not occur. In particular, we must have that $2a\omega_1 = \frac{3}{4}a^3$ or $\omega_1 = \frac{3}{8}a^2$.

f) We are left to solve

$$x_1'' + x_1 = -\frac{1}{4}a^3\cos(3\tau).$$

My method of undetermined coefficients we know that a solution will have the form

$$x_1(\tau) = A\cos(\tau) + B\sin(\tau) + C\cos(3\tau) + D\sin(3\tau),$$

where C and D help give the right hand side and A and B jelp us satisfy the initial conditions. In particular since

$$x_1'' + x_1 = -8C\cos(3\tau) - 8D\sin(3\tau) = -\frac{1}{4}a^3\cos(3\tau),$$

we have that $D = 0$ and $C = \frac{1}{32}a^3$. Then to satisfy the initial conditions that $x_1(0) = 0$ we need $A = -\frac{1}{32}a^3$ and to have $x_1'(0) = 0$ we need $B = 0$. So our solution is

$$x_1 = \frac{1}{32}a^3\big(\cos(3\tau) - \cos(\tau)\big).$$

7.6.22 We proceed much as in the previous problem. So we have that $\ddot{x} + x + \varepsilon x^2 = 0$ and we define a new time period $\tau = \omega t$ so that our equation becomes $\omega^2 x'' + x + \varepsilon x^2 = 0$. We now let $x(\tau,\varepsilon) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + O(\varepsilon^3)$ and $\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + O(\varepsilon^3)$. We now have that the first few terms are:

$$\begin{array}{rcllll}
\omega^2 x'' & \approx & x_0''(\tau) & +\varepsilon\big(x_1''(\tau) + 2\omega_1 x_0''(\tau)\big) & +\varepsilon^2\big(x_2''(t) + 2\omega_1 x_1''(\tau) + (\omega_1^2 + \omega_2)x_0(\tau)\big) \\
x & \approx & x_0(\tau) & +\varepsilon x_1(\tau) & +\varepsilon^2 x_2(\tau) \\
\varepsilon x^2 & \approx & & +\varepsilon(x_0(\tau))^2 & +\varepsilon^2 2x_0(\tau)x_1(\tau)
\end{array}$$

So this implies (by gathering terms according to powers of ε) that

$$\begin{array}{rcl}
x_0''(\tau) + x_0(\tau) & = & 0, \\
x_1''(\tau) + x_1(\tau) & = & -2\omega_1 x_0''(\tau) - (x_0(\tau))^2, \\
x_2''(\tau) + x_2(\tau) & = & -2\omega_1 x_1''(\tau) - (\omega_1^2 + \omega_2)x_0(t) - 2x_0(t)x_1(t).
\end{array}$$

The initial conditions translate into $x_0(0) = a$ and $x_k(0) = 0$ for $k \geq 1$ while $x_k'(0) = 0$ for all $k \geq 0$. So it is easy to solve (as in the previous case) that $x_0(\tau) = a\cos(\tau)$. Now putting this in to the next set of equations we have

$$x_1''(\tau) + x_1(\tau) = 2a\omega_1\cos(t) - a^2\cos^2(\tau) = 2a\omega_1\cos(\tau) - \frac{a^2}{2} - \frac{a^2}{2}\cos(2\tau).$$

(In the last step we used that $\cos^2(x) = (1 + \cos(2x))/2$.) Since we want to avoid the $\cos(\tau)$ term (i.e., to avoid secular terms) we would choose $\omega_1 = 0$. So we can now use the method of undetermined coefficients to see that our solution will be of the form

$$x_1(\tau) = A\cos(\tau) + B\sin(\tau) + C\cos(2\tau) + D\sin(2\tau) + E,$$

where C, D and E generate the particular solution while A and B are used to handle the initial conditions. Solving for C, D and E we have that

$$-3C\cos(2\tau) - 3D\sin(2\tau) + E = -\frac{a^2}{2} - \frac{a^2}{2}\cos(2\tau)$$

so that we need $C = a^2/6$, $D = 0$ and $E = -a^2/2$. Once we have this then the initial conditions tell us that we need $A+C+E = 0$ so that $A = a^2/3$ and similarly we need $B = 0$. So we have that

$$x_1(\tau) = \frac{a^2}{3}\cos(\tau) + \frac{a^2}{6}\cos(2\tau) - \frac{a^2}{2} = \frac{a^2}{6}\big(2\cos(\tau) + \cos(2\tau) - 3\big).$$

To avoid secular terms occurring in x_2 we would need that the following has no $\cos(\tau)$ or $\sin(\tau)$ terms

$$\begin{aligned}
&-2\omega_1 x_1''(\tau) - (\omega_1^2 + \omega)x_0(t) - 2x_0(t)x_1(t)\\
&\quad= -a\omega_2\cos(\tau) - \frac{a^3}{6}\cos(\tau)\big(2\cos(\tau) + \cos(2\tau) - 3\big)\\
&\quad= -a\omega_2\cos(\tau) - \frac{a^3}{3}\cos^2(\tau) - \frac{a^3}{6}\cos(\tau)\cos(2\tau) + \frac{a^3}{2}\cos(\tau)\\
&\quad= -a\omega_2\cos(\tau) - \frac{a^3}{6}\big(1 - \cos(2\tau)\big) - \frac{a^3}{12}\big(\cos(3\tau) + \cos(\tau)\big) + \frac{a^3}{2}\cos(\tau)
\end{aligned}$$

In particular to avoid secular terms we would need the coefficient of $\cos(\tau)$ to be 0. Gathering coefficients we need $-a\omega_2 - a^3/12 + a^3/2 = 0$, or rearranging, $a\omega_2 = (5/12)a^3$, and so we must choose $\omega_2 = (5/12)a^2$.

We now have that the period is $\omega = 1 + \frac{5}{12}a^2\varepsilon^2 + O(\varepsilon^3)$. We also have that

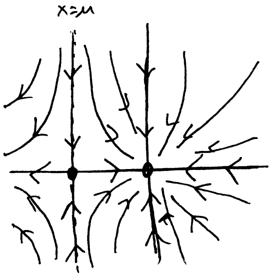
$$x(\tau) = a\cos(\tau) + \varepsilon\frac{a^2}{6}\big(2\cos(\tau) + \cos(2\tau) - 3\big) + O(\varepsilon^2),$$

which if we now use $\tau = \omega t$ becomes

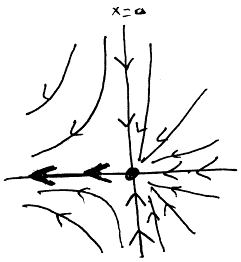
$$x(t) = a\cos(\omega t) + \varepsilon\frac{a^2}{6}\big(2\cos(\omega t) + \cos(2\omega t) - 3\big) + O(\varepsilon^2).$$

Finally, the center of oscillation will correspond to the constant term of this trigonometric polynomial which examining $x(t)$ is $\approx -\varepsilon a^2/2$.

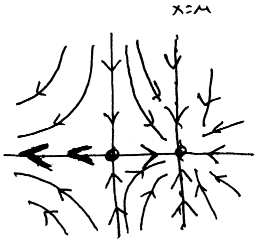
8.1.1 a) Let us first see where our bifurcation should occur. Calculating we have that our zeroes are at $x = 0$ and $x = \mu$ and so as μ passes through 0 we have fixed points colliding and so we expect our bifurcation to happen at 0. So below we have sketched $\mu < 0$, $\mu = 0$ and $\mu > 0$.



$\mu<0$

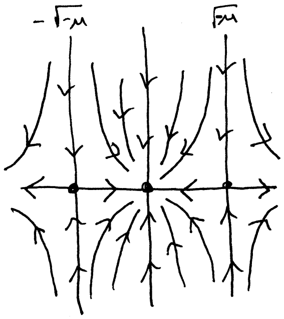


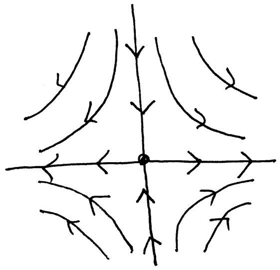
$\mu=0$



$\mu>0$

b) Again let us first see where our bifurcation should occur. Calculating the zeroes they occur at $x = 0$ and at $x = \pm\sqrt{-\mu}$, in the latter case these only exist for $\mu < 0$. So we go from 3 fixed points to 1 fixed point when we pass through $\mu = 0$ and so that is where we look for our bifurcation. So below we have sketched $\mu < 0$, $\mu = 0$ and $\mu > 0$.





8.1.8 Parameterizing (changing to x instead of ϕ) by letting $\dot{x} = y$ we have that $\dot{y} = \ddot{x} = \frac{1}{\varepsilon}(-y - \sin(x) + \gamma \sin(x)\cos(x))$. The fixed point will occur when $y = 0$ and $\sin(x)(1 - \gamma\cos(x)) = 0$, so either $x = 0$, $x = \pi$ and any solution to $\cos(x) = 1/\gamma$ (which will have no solutions for $\gamma < 1$, one solution for $\gamma = 1$ and two solutions for $\gamma > 1$). Taking the Jacobian we have

$$\begin{pmatrix} 0 & 1 \\ \frac{1}{\varepsilon}\big(-\cos(x) + \gamma\cos(2x)\big) & -\frac{1}{\varepsilon} \end{pmatrix}.$$

Looking at the determinant we have $\frac{1}{\varepsilon}\big(\cos(x) - \gamma\cos(2x)\big)$. For $x = \pi$ this is $(-1 - \gamma)/\varepsilon < 0$ and so this is always a saddle point. For $x = 0$ this is $(1-\gamma)/\varepsilon$ this is positive for $\gamma < 1$ (so a stable point) and is negative for $\gamma > 1$ (so a saddle). Since at $\gamma = 1$ we have the emergence of two additional fixed points at $\pm \arccos(\frac{1}{\gamma})$ (which are very close to 0). At those fixed points we have $\cos(x) = \frac{1}{\gamma}$ and $\cos(2x) = 2\cos^2(x) - 1 = \frac{2-\gamma^2}{\gamma^2}$ so that the determinant of the Jacobian is $\frac{1}{\varepsilon\gamma}(\gamma^2 - 1) > 0$ so that these fixed points are always stable.

Comparing this to the situation in Example 8.1.3 we see that we have a supercritical pitchfork bifurcation that occurs at $\gamma = 1$.

8.1.11 Since $\dot{v} = uv^2 - (a+k)v = v(uv - (a+k))$ then at a fixed point we must have that either $v = 0$ or $uv = a + k$. In the first case it follows that $u = 1$, in the second case we can substitute this into $(uv)v = a(1-u)$ to get that $v = \frac{a}{a+k}(1-u)$, in particular we have that at a fixed point

$$u\big(\frac{a}{a+k}(1-u)\big) = a + k \qquad \text{or} \qquad u(1-u) = \underbrace{\frac{(a+k)^2}{a}}_{=\gamma}.$$

Solving we have $u^2 - u - \gamma = 0$ so that $u = (1 \pm \sqrt{1-4\gamma})/2$, we also have that $v = \frac{a}{a+k}(1-u) = \frac{a}{a+k}(-1 \mp \sqrt{1-4\gamma})/2$.

Now that we know where our critical point is we need to find when we have a bifurcation. This will happen when our determinant of the Jacobian changes sign (since we have no gotten to Hopf bifurcations yet). Looking at the Jacobian we have

$$\begin{pmatrix} -a - v^2 & -2uv \\ v^2 & 2uv - (a+k) \end{pmatrix}.$$

At the point $u = 1$, $v = 0$ we see that the eigenvalues are always negative and so there is no bifurcation. Now let us consider the other point. Namely we have $uv = a + k$ and so

$$\left|\begin{pmatrix} -a - v^2 & -2uv \\ v^2 & 2uv - (a+k) \end{pmatrix}\right| = \left|\begin{pmatrix} -a - v^2 & -2(a+k) \\ v^2 & a+k \end{pmatrix}\right|$$
$$= -(a+v^2)(a+k) + 2v^2(a+k) = (a+k)(v^2 - a).$$

So the problem reduces down to determining when $v^2 = a$ (if $v^2 > a$ then we have a node while if $v^2 < a$ we have a saddle). So plugging in what we know for v we have

$$v^2 = \frac{a^2}{(a+k)^2}\big(\frac{-1 \mp \sqrt{1-4\gamma}}{2}\big)^2 = a \qquad \text{or} \qquad 1 \mp 2\sqrt{1-4\gamma} + (1-4\gamma) = 4\gamma,$$

which after more rearranging becomes

$$4\gamma - 1 = \mp\sqrt{1-4\gamma} \qquad \text{or} \qquad 16\gamma^2 - 8\gamma + 1 = 1 - 4\gamma.$$

This last expression can be solved explicitly for γ, namely either $\gamma = 0$ (which is impossible given our constraints), or $\gamma = \frac{1}{4}$. We now have that the bifurcation occurs when

$$\gamma = \frac{(a+k)^2}{a} = \frac{1}{4} \qquad \text{or} \qquad 4(a+k)^2 = a \qquad \text{or} \qquad 2(a+k) = \pm\sqrt{a}.$$

This last expression can easily be solved for k to get $k = -a \pm \frac{1}{2}\sqrt{a}$.

8.2.1 From the last homework assignment (7.5.6a) we know that we can rewrite this system as $\dot{x} = \mu(y - (\frac{1}{3}x^3 - x))$ and $\dot{y} = (a - x)/\mu$. Further the unique fixed point is as $(a, F(a))$ with the Jacobian

$$\begin{pmatrix} -\mu(a^2-1) & \mu \\ -1/\mu & 0 \end{pmatrix}.$$

The characteristic equation is then $\lambda^2 + \mu(a^2-1)\lambda + 1 = 0$. In particular the eigenvalues are

$$\frac{\mu(1-a^2) \pm \sqrt{\mu^2(1-a^2)^2 - 4}}{2}.$$

So what happens with the eigenvalues is roughly along the lines of the following:

- $0 < a < \sqrt{1-(2/\mu)}$: In this case both eigenvalues are positive, one is very small (close to 0) and one is very large (or order μ). As a gets closer to $\sqrt{1-(2/\mu)}$ the eigenvalues come closer together.
- $a = \sqrt{1-(2/\mu)}$: In this case we have a repeated (positive) eigenvalue of 1.
- $\sqrt{1-(2/\mu)} < a < 1$: We now have two complex eigenvalues with positive real part, as a gets closer to 1 the real part gets closer to 0.
- $a = 1$: We have eigenvalues of $\pm i$. This is the location of the Hopf bifurcation.
- $1 < a < \sqrt{1+(2/\mu)}$: We continue to have two complex eigenvalues, now with negative real part, as a gets closer to $\sqrt{1+(2/\mu)}$ the real part gets closer to -1 and the complex part goes to 0.
- $a = \sqrt{1+(2/\mu)}$: In this case we have a repeated (negative) eigenvalue of -1.
- $\sqrt{1+(2/\mu)} < a < \infty$: In this case both eigenvalues are negative, one will become very small (close to 0) and the other very large (or order $-\mu$).

8.2.12 a) We have that $\omega = 1$, $f(x,y) = xy^2$ and $g(x,y) = -x^2$. In particular we have that $f_{xyy} = 2$ and $f_{xxx} = g_{xxy} = g_{yyy} = g_{xy} = g_{yy} = f_{xx} = 0$. Since $f_{xy} = 2y$ which will evaluate to 0 at the origin. Putting all these in, we have that $16a = 2$, or $a = 1/8 > 0$.

b) Since a is positive we have a subcritical Hopf bifurcation at $(0,0)$.