Homework 9 solutions

8.1.12 a) The fixed points will occur when $\dot{\theta}_1 = 0$ and $\dot{\theta}_2 = 0$. This implies

$$\sin(\theta_1) = K\sin(\theta_1 - \theta_2) = -\sin(\theta_2).$$

This implies that either $\theta_2 = -\theta_1$ or $\theta_2 = \pi + \theta_1$. The latter case would imply that $\sin(\theta_1) = \sin(\theta_2) = 0$ so that either $\theta_1 = 0$ and $\theta_2 = \pi$ or $\theta_1 = \pi$ and $\theta_2 = 0$. The Jacobian of the system is

$$J(\theta_1, \theta_2) = \begin{pmatrix} K\cos(\theta_1 - \theta_2) - \cos(\theta_1) & -K\cos(\theta_1 - \theta_2) \\ -K\cos(\theta_2 - \theta_1) & K\cos(\theta_2 - \theta_1) - \cos(\theta_2) \end{pmatrix},$$

so that $J(0, \pi) = \left(\begin{smallmatrix} -K-1 & K \\ K & -K+1 \end{smallmatrix}\right)$ and $J(\pi, 0) = \left(\begin{smallmatrix} -K+1 & K \\ K & -K-1 \end{smallmatrix}\right)$ both of which have a determinant of -1, and so these are *saddle* points.

Now let us consider the case $\theta_2 = -\theta_1$. Then putting this in we see that we have

$$0 = K\sin(2\theta_1) - \sin(\theta_1) = 2K\sin(\theta_1)\cos(\theta_1) - \sin(\theta_1) = \sin(\theta_1)\big(2K\cos(\theta_1) - 1\big).$$

This implies that we either have $\sin(\theta_1) = 0$ (giving that $\theta_1 = \theta_2 = 0$ or $\theta_1 = -\theta_2 = \pi$), or $\theta_1 = \pm\arccos(1/2K)$ (assuming $2K \geq 1$). In the first case we have $J(0, 0) = \left(\begin{smallmatrix} K-1 & -K \\ -K & K-1 \end{smallmatrix}\right)$, this has a determinant of $1 - 2K$ and so is a *saddle* for $2K > 1$ and is a stable node for $2K < 1$ (i.e., positive determinant and negative trace, note that because the matrix $J(\theta_1, \theta_2)$ is symmetric all eigenvalues will be real so no spirals are possible). In the second case we have $J(\pi, -\pi) = \left(\begin{smallmatrix} K+1 & -K \\ -K & K+1 \end{smallmatrix}\right)$, this has a determinant of $1 + 2k$ and a trace of $2K + 2$ and so both of the eigenvalues are positive showing that this is an *unstable* node.

This leaves us with the most interesting case of $\theta_1 = -\theta_2 = \arccos(1/2K)$. In order to evaluate this we first note that $\cos(\theta_1) = \cos(\theta_2) = 1/2K$. And then again note that $\cos(\theta_1 - \theta_2) = \cos(2\theta_1) = 2\cos^2(\theta_1) - 1 = (1/2K^2) - 1$. Therefore we have

$$\begin{aligned} J\big(\arccos(1/2K), -\arccos(1/2K)\big) &= \begin{pmatrix} K\big(\frac{1}{2K^2} - 1\big) - \frac{1}{2K} & -K\big(\frac{1}{2K^2} - 1\big) \\ -K\big(\frac{1}{2K^2} - 1\big) & K\big(\frac{1}{2K^2} - 1\big) - \frac{1}{2K} \end{pmatrix} \\ &= \begin{pmatrix} -K & K - \frac{1}{2K} \\ K - \frac{1}{2K} & -K \end{pmatrix}. \end{aligned}$$

This has a trace of $-2K < 0$ and a determinant of $1 - \frac{1}{4K^2}$, which using our assumption that $2K > 1$ (i.w., $4K^2 > 1$) implies that the determinant is positive. In particular these points will be *stable* nodes.

b) Looking at the word done in part (a) we see that at $K = 1/2$ that one stable node splits into two stable nodes and a saddle point. So this will be a supercritical pitchfork bifurcation.

c) To be gradient we would need $\partial V/\partial\theta_1 = -K\sin(\theta_1-\theta_2)+\sin(\theta_1)$, which would imply (upon integrating) that $V = K\cos(\theta_1-\theta_2) - \cos(\theta_1) + f(\theta_2)$ for some function f. Now differentiating with respect to θ_2 we see that we need $\partial V/\partial\theta_2 = K\sin(\theta_1-\theta_2)+f'(\theta_2) = -K\sin(\theta_2-\theta_1)+\sin(\theta_2)$ (recalling that sine is an odd function). So we need $f'(\theta_2)=\sin(\theta_2)$ so that we can choose $f(\theta_2) = -\cos(\theta_2)$. so we have a gradient system by setting $V = K\cos(\theta_1-\theta_2)-\cos(\theta_1)-\cos(\theta_2)$.

d) By Theorem 7.2.1 we know that no gradient system has a closed orbit and since part (c) shows that this system is a gradient system there cannot be a closed orbit.

8.3.1 a) From $\dot{y} = 0$ we have $ax^2y = bx$ which if we substitute into $\dot{x} = 0$ gives us $1 - bx + x + bx = 0$ and so $x = 1$, this in turn implies that $y = b/a$. So there is a unique fixed point at $(x^*, y^*) = (1, b/a)$.

The Jacobian of this system is

$$J(x,y) = \begin{pmatrix} -(b+1)+2axy & ax^2 \\ b-2axy & -ax^2 \end{pmatrix}, \quad \text{so} \quad J\big(1,\frac{b}{a}\big) = \begin{pmatrix} b-1 & a \\ -b & -a \end{pmatrix}.$$

This has determinant $-ab+a+ab = a > 0$, while the trace is $b-a-1$ in particular when $b < a+1$ then the trace is negative and the point is a sink (possibly a spiral) while when $b > a+1$ the trace is positive the point is a source (possibly a spiral).

c) From part (a) we know that a bifurcation (i.e., change in behavior) occurs when $b = a+1$.

d) A limit cycle must exist when the interior fixed point is repelling (i.e., then we can get a trapping region without fixed points which is invariant by punching out a small hole around the fixed point). In our case this happens when the eigenvalues have positive real part and as we saw in part (a) this corresponds to that case that $b > a+1 = b_c$.

e) When $b \approx b_c$ we have that the eigenvalues are

$$\frac{(b-a-1) \pm \sqrt{(b-a-1)^2-4a}}{2a} \approx \pm i\sqrt{a}.$$

In particular we have that the limit cycle has frequency approximately $\sqrt{a}$, i.e., so the period is approximately $2\pi/\sqrt{a}$.

9.2.3 We need to show that $W = x^2+y^2+(z-r-\sigma)^2$ decreases along trajectories if we are *sufficiently* far out (which will be determined how far is sufficient, this is very similar with what the book does on page 315). So computing we have that

$$\begin{aligned}\frac{d}{dt}\big(x^2+y^2+(z-r-\sigma)^2\big) &= 2x\dot{x}+2y\dot{y}+2(z-r-\sigma)\dot{z}\\ &= 2x\sigma(y-x)+2y(rx-y-xz)+2(z-r-\sigma)(xy-bz)\\ &= -2\sigma x^2-2y^2-2bz^2+2rbz+2\sigma bz\\ &= -2\sigma x^2-2y^2-2b\big(z-\frac{\sigma+r}{2}\big)^2+2b\big(\frac{\sigma+r}{2}\big)^2.\end{aligned}$$

In particular we will have that W is decreasing along all trajectories when we are *outside* of the ellipse

$$2\sigma x^2+2y^2+2b\big(z-\frac{\sigma+r}{2}\big)^2=2b\big(\frac{\sigma+r}{2}\big)^2.$$

Now it suffices to pick C large enough so that the ellipsoid is inside the sphere $C = x^2+y^2+(z-r-\sigma)^2$, one such C that will work is $C=\big(1+\frac{b}{4}+\frac{b}{4\sigma}\big)(\sigma+r)^2$.

9.2.6 a) If V represents volume then we have that $\dot{V}=\int_V \nabla\cdot\mathbf{f}\,dV$. In our case we have that

$$\nabla\cdot\mathbf{f}=\frac{\partial}{\partial x}\big(-vx+zy\big)+\frac{\partial}{\partial y}\big(-vy+(z-a)x\big)+\frac{\partial}{\partial z}\big(1-xy\big)=-2v<0.$$

In particular we have that $\dot{V}=-2vV$ so that $V=Ce^{-2vt}$, i.e., that volumes shrink exponentially fast.

b) From $\dot{z}=0$ we know that $xy=1$, so that $x\neq 0$ and $y\neq 0$. So now let $x=\pm k$ (for some k to be determined), then we have that $y=1/x=\pm k^{-1}$. From $\dot{x}=0$ we have that $z=vx/y=vk^2$ while from the second equation we have that $z=a+vy/x=a+vk^{-2}$. So in particular we must have that $vk^2=a+vk^{-2}$ or $v(k^2-k^{-2})=a$.

c) We have that the Jacobian is

$$J(x,y,z)=\begin{pmatrix}-v & z & y\\ z-a & -v & x\\ -y & -x & 0\end{pmatrix}.$$

Note from above we have that $z-a=vy/x=vk^{-2}$. So we now have

$$J(k,k^{-1},vk^2)=\begin{pmatrix}-v & vk^2 & k^{-1}\\ vk^{-2} & -v & k\\ -k^{-1} & -k & 0\end{pmatrix}$$

The characteristic polynomial of this matrix is

$$\lambda^3+2v\lambda^2+\big(k^2+\frac{1}{k^2}\big)\lambda+2v\big(k^2+\frac{1}{k^2}\big)$$

Note that this can be rewritten in the form

$$\lambda^2(\lambda + 2v) + \big(k^2 + \frac{1}{k^2}\big)(\lambda + 2v) = (\lambda + 2v)(\lambda^2 + \big(k^2 + \frac{1}{k^2}\big)).$$

So the roots are $-2v$ and $\pm i\sqrt{k^2 + k^{-2}}$. So in one direction we are compressing (at an exponential rate of $-2v$, compare part (a)) and then in the other directions we are spinning.

Since $J(-k, -k^{-1}, vk^2)$ also has the same characteristic equation is also has the same eigenvalues and hence the same behavior.