Midterm 1 solutions

(1) Consider the system

$$\dot{x} = y, \qquad \dot{y} = 4x^3 - 8x.$$

(a) Show that it is conservative and find an energy function.

Solution: We have that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4x^3 - 8x}{y} \qquad \text{or} \qquad y\,dy = (4x^3 - 8x)\,dx.$$

Integrating both sides we have that $\frac{1}{2}y^2 = x^4 - 4x^2 + C$ or

$$C = \underbrace{\frac{1}{2}y^2 - x^4 + 4x^2}_{= \text{ energy function}}.$$

(b) Find the fixed points and determine what kind each one is.

Solution: Fixed points occur when $y = 0$ and $4x^2 - 8x = 0$ or $4x(x^2 - 2) = 0$. So the fixed points are $(0, 0)$ and $(\pm\sqrt{2}, 0)$. To determine stability we look at the Jacobian which is $\left(\begin{smallmatrix} 0 & 1 \\ 12x^2-8 & 0 \end{smallmatrix}\right)$. At $(\pm\sqrt{2}, 0)$ the Jacobian becomes $\left(\begin{smallmatrix} 0 & 1 \\ 16 & 0 \end{smallmatrix}\right)$, which has a negative determinant. In particular we have that $(\pm\sqrt{2}, 0)$ are *saddle* points. At $(0, 0)$ the Jacobian becomes $\left(\begin{smallmatrix} 0 & 1 \\ -8 & 0 \end{smallmatrix}\right)$ which has a positive determinant and a trace of zero. In particular the eigenvalues are the roots of $\lambda^2 + 8 = 0$ which are $\pm 2\sqrt{2}\,i$; so this implies that $(0, 0)$ is a *center*.

(c) Find the period of small oscillations around the fixed point which is a center.

Solution: From part (b) we know that the eigenvalues around the center are $\pm 2\sqrt{2}\,i$. So the period is $2\pi/2\sqrt{2} = \pi/\sqrt{2}$.

(d) This system has heteroclinic trajectories: trajectories which approach one saddle point as $t \to -\infty$ and a different one as $t \to +\infty$. Find the energy of such a trajectory and its equation in the xy plane, e.g. in the form $y = f(x)$.

Solution: Plugging the points into the energy function (found in part (a)) we have that the energy along the trajectory which hits the saddle is $\frac{1}{2}(0)^2 - (\pm\sqrt{2})^4 + 4(\pm\sqrt{2})^2 = 0 - 4 + 8 = 4$. So we have that the desired relationship is

$$4 = \frac{1}{2}y^2 - x^4 + 4x^2, \qquad \text{or rearranging} \qquad y = \pm\sqrt{8 - 8x^2 + 2x^4}\,.$$

(2) Consider the system

$$\dot{x} = x(9 - x^2 - 8y), \qquad \dot{y} = y(2 - x - y).$$

This is a type of competititce species model, so assume $x, y \geq 0$. The fixed points in the first quadrant are $(0,0)$, $(0,2)$, $(3,0)$, and $(1,1)$.

(a) Use linearization to determine the type of each fixed point.

SOLUTION: To determine the stability we again will use the Jacobian which in this case is

$$J(x,y) = \begin{pmatrix} 9 - 3x^2 - 8y & -8x \\ -y & 2 - x - 2y \end{pmatrix}.$$

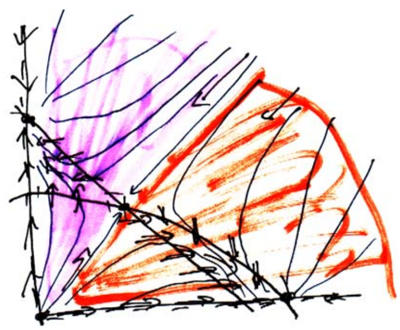
We have that $J(0,0) = \left(\begin{smallmatrix} 9 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ which has two positive eigenvalues and so this is an *unstable* node. Next, we have that $J(0,2) = \left(\begin{smallmatrix} -7 & 0 \\ -2 & -2 \end{smallmatrix}\right)$ which has two negative eigenvalues and so this is a *stable* node. Next, we have that $J(3,0) = \left(\begin{smallmatrix} -18 & -24 \\ 0 & -1 \end{smallmatrix}\right)$ which also has two negative eigenvalues and so this is also a *stable* node. Finally, we have that $J(1,1) = \left(\begin{smallmatrix} -2 & -8 \\ -1 & -1 \end{smallmatrix}\right)$ which has a determinant of -6 and so this must be a *saddle* point.

(b) Use inted theory to either show that there are no closed orbits, or else determine where a possible closed orbit might be.

SOLUTION: There are no closed orbits. We can rule out hitting the axes since we can explicitly find these trajectories. Also, a closed orbit must enclose at least one fixed point, which only leaves us with $(1,1)$. But this is a saddle which has index -1 so this also does not work.

(3) Sketch a phase portrait for the system in problem 2. If there are stable fixed point, describe the basin of attraction of each.

SOLUTION: To sketch this we can draw the nullclines and use these as a guide. A picture is shown below where we have drawn the nullclines, then sketched a few trajectories. Two trajectories (one coming from the origin and the other from "infinity") will head toward the saddle at $(1,1)$. Otherwise trajectories in the first quadrant head to one of the two stable nodes located at $(0,2)$ and $(3,0)$. We have shaded the two basins of attraction in the picture below.



(4) Consider the system

$$\dot{x} = y(3x + 3 - y^2), \qquad \dot{y} = x.$$

(a) Show that the system is reversible. What does this imply about the phase portrait? Be as specific as possible.

SOLUTION: Starting with our system

$$\begin{aligned} \dot{x} &= y(4x + 4 - y^2) \\ \dot{y} &= x \end{aligned}$$

let us now replace t with $-t$ and replace y by $-y$. Doing this we have

$$\begin{aligned} -\dot{x} &= -y(4x + 4 - (-y)^2) \\ -(-\dot{y}) &= x \end{aligned}$$

which simplifies to the original system. This shows that the system is reversible and more particularly shows that the phase portrait is *symmetric about the x-axis*.

(b) Show that the system is *not* conservative. Hint: what types of fixed points does it have?

SOLUTION: Following the hint, it will suffice to show that there is a stable or unstable node (i.e., conservative systems *cannot* have a stable or unstable node, so if there were one in this system if could not be conservative). To start we first note that at a fixed point we must have that $x = 0$ and $y(4x + 4 - y^2) = 0$. So the fixed points for this system are $(0, 0)$ and $(0, \pm 2)$. We also have that the Jacobian is

$$J(x, y) = \begin{pmatrix} 4y & 4x + 4 - 3y^2 \\ 1 & 0 \end{pmatrix}.$$

At $(0, 0)$ we have $J(0, 0) = \left(\begin{smallmatrix} 0 & 4 \\ 1 & 0 \end{smallmatrix}\right)$ which has a negative determinant and so is a *saddle* node (which is perfectly OK). So going on we have at $(0, 2)$ that

$J(0,2) = \begin{pmatrix} 8 & -8 \\ 1 & 0 \end{pmatrix}$ which has as its eigenvalues $\lambda = 4 \pm 2\sqrt{2} > 0$, so this is an *unstable* node and so we can now conclude the system is *not* conservative. Going on we have at $(0,-2)$ that $J(0,-2) = \begin{pmatrix} -8 & -1 \\ 1 & 0 \end{pmatrix}$ which has as its eigenvalues $\lambda = -4 \pm 2\sqrt{2} < 0$ which is a *stable* node.

(Note that the stable node is directly opposite across the x-axis from the unstable node. In light of part (a) this was to be expected.)